SOLUTION OF BERNSTEIN'S APPROXIMATION PROBLEM

HARRY POLLARD

In his famous monograph on approximation theory [2], S. Bernstein initiated the study of the closure properties of sets of functions \( \{u^nK(u)\}_{n=0}^\infty \) on the real line. It is supposed that \( K(u) \) is continuous on \((-\infty, \infty)\) and that \( u^nK(u) \) vanishes at \( u = \pm \infty \) for each value of \( n \). The problem is to decide when the set \( \{u^nK(u)\} \) is fundamental in the space \( C_0 \) of functions continuous on \((-\infty, \infty)\), vanishing at \( \pm \infty \), and normed by \( \|f\| = \max |f(u)| \). So far no necessary and sufficient conditions have been given. A recent paper of Carleson [3] reviews most of the known results, but the paper [1] which seems to come closest to the true conditions has been overlooked.

It is the purpose of this note to give a complete solution. It applies to either real- or complex-valued functions and may be read either way.

THEOREM. In order that \( \{u^nK(u)\}_{n=0}^\infty \) be fundamental in \( C_0 \) it is necessary and sufficient that

\[
(1) \quad K(u) \neq 0, \quad -\infty < u < \infty;
\]

\[
(2) \quad \int_{-\infty}^{\infty} \frac{\log |K(u)|}{1 + u^2} \, du = -\infty;
\]

and that there exists a sequence of polynomials \( p_n \) such that

\[
(3) \quad \lim_{n \to \infty} p_n(u)K(u) = 1; \quad |p_n(u)K(u)| \leq C, \quad -\infty < u < \infty.
\]

1. The necessity. The necessity of (1) is obvious and of (2) is well known [1; 3]. To prove the necessity of the remaining conditions let \( 0_n(u) \) denote the continuous function which is unity on \((-n, n)\), vanishes outside \((-n-1, n+1)\), and is linear in the remaining in-

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tervals. Since \( \{u^*K(u)\} \) is fundamental there exists for each \( n \) a polynomial \( p_n \) such that
\[
| p_n(u)K(u) - 0_n(u) | \leq 2^{-n}.
\]
Now let \( n \to \infty \) and (3) follows.

2. A lemma. To prove the sufficiency we shall need the following result.

**Lemma.** Let \( \alpha(x) \) be of bounded variation on \((-\infty, \infty)\). Then the functions
\[
F_\pm(x) = \alpha'(x) \pm \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(u)}{x - u}
\]
exist almost everywhere when the integral is interpreted as a principal value. Moreover
\[
(2.1) \quad \int_{-\infty}^{\infty} \frac{\log |F_\pm(z)|}{1 + x^2} dz < \infty
\]
for at least one choice of the \( \pm \) sign, unless \( \alpha \) is substantially a constant.

The first part of the theorem follows from a result of Loomis [4] on Hilbert transforms. Note that (2.1) is the same for either choice of sign if \( \alpha \) is real, so that the complication comes from the possibility that it is complex-valued.

To establish (2.1) consider the function
\[
H(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(u)}{z - u}, \quad z = x + iy.
\]
\( H(z) \) is analytic for \( y > 0 \) and for \( y < 0 \). It cannot be identically zero in both half-planes unless \( \alpha \) is substantially a constant. Ruling out this case, we may assume \( H \neq 0 \) in one of these half-planes, say \( y > 0 \). We shall establish (2.1) with the + sign.

Now \( H = U+iV \), where \( U(x, y) \) and \( V(x, y) \) are defined by
\[
U(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(u)}{(x - u)^2 + y^2}
\]
and
\[
V(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - u)d\alpha(u)}{(x - u)^2 + y^2}.
\]
Since
it follows from Schwarz's inequality that
\[ \int_{-\infty}^{\infty} \left| \frac{U(x, y)}{1 + x^2} \right| dx \leq C < \infty. \]

As for \( V(x, y) \), an argument used by Titchmarsh [6, pp. 144–145] shows that
\[ \int_{-\infty}^{\infty} \left| \frac{V(x, y)}{1 + x^2} \right| dx \leq C < \infty. \]
(Titchmarsh proves this when \( \alpha \) is an integral, but his argument is quite general.) Consequently
\[ \int_{-\infty}^{\infty} \left| \frac{H(x + iy)}{1 + x^2} \right| dx \leq C < \infty. \]

Map the half-plane \( \Im z > 0 \) into the unit circle \( |w| < 1 \) by \( z = i(1 - w)/(1 + w) \). If we write \( w = re^{\theta}, \ h(w) = H(z) \), then \( d\theta = 2(1 + x^2)^{-1} dx \) and the preceding formula becomes
\[ \int_{0}^{2\pi} \left| h(re^{i\theta}) \right|^{1/2} d\theta \leq C < \infty, \quad 0 \leq r < 1. \]

A standard argument (see, for example, [5, pp. 19–20]) shows that
\[ \int_{0}^{2\pi} \left| \log \left| h(re^{i\theta}) \right| \right| d\theta \leq C < \infty. \]

Since \( h \) is of class \( H^{1/2} \) the limit \( h(e^{i\theta}) = \lim_{r \to 1} h(re^{i\theta}) \) exists almost everywhere. Hence by Fatou’s lemma
\[ \int_{0}^{2\pi} \left| \log \left| h(e^{i\theta}) \right| \right| d\theta \leq C < \infty. \]

Mapping back, we get
\[ \int_{-\infty}^{\infty} \left| \frac{\log \left| H(x + i0) \right|}{1 + x^2} \right| dx < \infty. \]

It remains only to identify \( H(x + i0) \) with \( F_{+}(x) \). This amounts to showing that almost everywhere
\[ \lim_{r \to 0^{+}} U(x, y) = \alpha'(x), \]
Each of these is well known if $\alpha$ is absolutely continuous [6, Chap. V]. It is therefore enough to prove them when $\alpha$ is singular, that is, when $\alpha'(x) = 0$ almost everywhere. We shall prove only the second, (2.2), the argument for the first being similar and easier. For simplicity in printing we also write "$y \to 0+$" for "$y \to 0$".

Let $x_0$ be a point for which $\int_{-\infty}^{\infty} \frac{d\alpha(u)}{u - x}$ exists and for which $\alpha'(x_0) = 0$. This is true for almost all $x_0$. By a change of variable we may assume $x_0 = 0$ and (2.2) becomes

$$
\lim_{y \to 0+} \int_{-\infty}^{\infty} \frac{u \alpha(u)}{u^2 + y^2} = \int_{-\infty}^{\infty} \frac{d\alpha(u)}{u}.
$$

Clearly it is enough to show that

$$
\lim_{y \to 0+} \int_{0}^{\infty} \frac{u \alpha(u)}{u^2 + y^2} - \int_{-\infty}^{0} \frac{d\alpha(u)}{u} = 0,
$$

We confine ourselves to (2.3).

In (2.3) replace $\alpha$ by $\beta = \alpha - \alpha(0)$ and integrate by parts. Since $\beta'(0) = 0$, (2.3) reduces to

$$
\lim_{y \to 0+} \left\{- \int_{0}^{\infty} \beta(u) \frac{u}{u^2 + y^2} du - \int_{y}^{\infty} \frac{\beta(u)}{u^2} du \right\} = 0.
$$

Because $\beta(u) = o(u)$, $u \to 0$, we have

$$
\int_{0}^{y} \beta(u) \frac{u}{u^2 + y^2} du = o(1), \quad y \to 0,
$$

and the problem is further reduced to showing that

$$
\int_{y}^{\infty} \beta(u) \left\{ \frac{u}{(u^2 + y^2)} + \frac{1}{u^3} \right\} du = o(1), \quad y \to 0.
$$

The last integral, after a change of variable, is

$$
\int_{1}^{\infty} \frac{\beta(yu)}{yu} \left\{ \frac{d}{du} \frac{u}{u^2 + 1} + \frac{1}{u^3} \right\} du,
$$

which is dominated by
Because $\beta$ is bounded, $\beta(0) = 0$, and $\beta'(0) = 0$, the expression $\beta(yu)/yu$ approaches zero boundedly on $1 \leq u < \infty$ as $y \to 0$. Consequently the preceding expression converges to zero with $y$, and the proof is complete.

3. The sufficiency. Assume that (1), (2), (3) hold. Suppose that

$$\int_{-\infty}^{\infty} u^n K(u) d\sigma(u) = 0, \quad n = 0, 1, \ldots,$$

where $\sigma$ is of bounded variation. We must show that $\sigma$ is substantially a constant.

If it is not we may form the function

$$s(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma(u)}{x - u}$$

and conclude from the lemma that for some choice of the $\pm$ sign

$$\int_{-\infty}^{\infty} \left| \log \left| \frac{\sigma'(x) \pm is(x)}{1 + x^2} \right| \right| dx < \infty.$$

Since $K(u) \neq 0$, a similar remark applies to the function

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{K(u) d\sigma(u)}{x - u}$$

and we have

$$\int_{-\infty}^{\infty} \left| \log \left| \frac{K(x) \sigma'(x) \pm ig(x)}{1 + x^2} \right| \right| dx < \infty.$$

It is important to know that we may choose the same sign in both (3.2) and (3.3). According to the proof of the lemma we can do this if the functions

$$S(z) = \int_{-\infty}^{\infty} \frac{d\sigma(u)}{z - u}, \quad G(z) = \int_{-\infty}^{\infty} \frac{K(u) d\sigma(u)}{z - u}$$

have a common half-plane, $\gamma > 0$ or $\gamma < 0$, in which neither is identically zero. The identity

$$\frac{1}{z - u} = \frac{1}{z} + \frac{u}{z^2} + \cdots + \frac{u^{n-1}}{z^n} + \frac{u^n}{z^n(z - u)}$$
and (3.1) enable us to rewrite $G(z)$ as

$$ G(z) = \frac{1}{z^n} \int_{-\infty}^{\infty} \frac{u^n K(u) d\sigma(u)}{z - u}. $$

Consequently for each polynomial $p_n$ of (3) we have

$$ p_n(z) G(z) = \int_{-\infty}^{\infty} \frac{p_n(u) K(u)}{z - u} d\sigma(u). $$

Since $z$ is not real, (3) enables us to conclude that

$$(3.5) \quad \lim_{n \to \infty} p_n(z) G(z) = S(z).$$

Now $S(z)$ is not identically zero in at least one of the half-planes, say $y > 0$. Hence, by (3.5), $G(z)$ cannot vanish there identically either. We may therefore assume that both (3.2) and (3.3) are valid with the + sign.

In the identity (3.4) replace $z$ by $x$. The resulting formula and (3.1) enable us to rewrite $g(x)$ as

$$ g(x) = \frac{1}{x^n} \int_{-\infty}^{\infty} \frac{u^n K(u) d\sigma(u)}{x - u}, $$

so that

$$ p_n(x) g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{p_n(u) K(u)}{x - u} d\sigma(u) $$

and

$$ p_n(x) g(x) - s(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{p_n(u) K(u) - 1}{x - u} d\sigma(u). $$

By another result of Loomis [4] the measure of the set for which $|p_n(x) g(x) - s(x)| > \epsilon$ is at most

$$ \frac{A}{\epsilon} \int_{-\infty}^{\infty} \left| p_n(u) K(u) - 1 \right| d\sigma(u), $$

where $A$ is an absolute constant. In view of (3) this approaches zero as $n \to \infty$. Hence $p_n(x) g(x)$ converges to $s(x)$ in measure, so that a subsequence converges almost everywhere to $s(x)$. By (3), $p_n(x)$ converges to $1/K(x)$. Therefore

$$ g(x) = K(x) s(x) $$

for almost all $x$. From this identity we obtain
K(x) = \frac{K(x)\sigma'(x) + ig(x)}{\sigma'(x) + is(x)}.

Note that by (3.2) and (3.3) neither the numerator nor the denominator can vanish on a set of positive measure. Moreover by these same results

\int_{-\infty}^{\infty} \frac{\log |K(x)|}{1 + x^2} \, dx < \infty,

which contradicts hypothesis (2).

Therefore \sigma must be substantially a constant, and the proof is complete.

References


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