MINIMAL SETS OF VISIBILITY

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Let $S$ be a set in an $n$-dimensional Euclidean space, $E_n$. The following concept was used by Horn and Valentine [2] in their study of $L$ sets, and it provides the basis of this investigation.

**Definition 1.** A set $V \subset S$ is a set of visibility in $S$ if, given any point $p \in S$, there exists a point $q \in V$ such that the closed segment $pq \subset S$.

**Notation.** Given a point $x \in S$, let $V(x)$ denote a continuum $^1$ of visibility in $S$ which contains $x$. The notation $V_1(x)$ will also be used.

**Definition 2.** The set $V(x)$ is a minimal continuum of visibility in $S$ relative to $x$ if, for any other continuum of visibility $V(x)$, we have $V(x) \subseteq V_1(x)$.

A corresponding definition holds if we replace the word "continuum" by the words "compact convex set."

It is our purpose to investigate sets for which $V(x)$ is unique for each $x \in S$. The most interesting result is contained in Theorem 2. The corresponding theory in which maximal convex sets are considered has been developed by Strauss and Valentine [3]. The two theories are decidedly different, and this difference is explained at the end of this article.

1. **Minimal compact connected sets of visibility.**

**Theorem 1.** Let $S$ be a closed set in $E_n$. Suppose each point $x \in S$ is contained in a unique minimal continuum of visibility $V(x)$ in $S$. Then either $S$ is convex or the product $\prod_{x \in S} V(x)$ is a nonempty continuum. (Both conclusions hold if and only if $S$ is a single point.)

**Proof.** In this and later proofs we denote the line joining $x$ and $y$ by $L(x, y)$.

Suppose there exists two sets $V(x)$ and $V(y)$ such that $V(x) \cdot V(y) = 0$. By Definition 1 there exists a point $q \in V(x)$ such that $yq \subset S$. Let $z$ be the point of $V(x) \cdot yq$ which is nearest to $y$. The uniqueness of $V(z)$ implies that $V(z) \subset V(x)$ and that $V(z) \subset yz + V(y)$. Since $V(x) \cdot V(y) = 0$, the uniqueness of $V(z)$ together with $V(x) \cdot yz = z$ imply that $V(z) = z$. Hence if $V(x) \cdot V(y) = 0$, $S$ is starlike $^2$ with respect to $z$.

Hence, if $V(x) \cdot V(y) = 0$, since $V(y)$ is unique, we have $V(y) = yu$

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$^1$ A continuum is a compact connected set.

$^2$ A set $S$ is starlike if there exists a point $x \in S$ such that $V(x) = x$. 

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Since from the uniqueness of $V(u)$ we have $V(u) \subset u$, and $V(u) \subset u y$, then $V(u) = u$. If $S \subset L(y, z)$, then clearly $S$ is convex. If $S \subset L(y, z)$, choose any point $w \in S - L(y, z)$. Since $V(x) = z$, $V(u) = u$, we have $V(w) \subset w$, $V(w) \subset w u$. Hence $V(w) = w$. By the same token if $p \in L(y, z) - S$, then $V(p) \subset p w$, $V(p) \subset p z$, whence $V(p) = p$. Thus for any point $a \in S$, we have $V(a) = a$. Hence, if $V(x) \cdot V(y) = 0$, the set $S$ is convex.

Now, assume $S$ is not convex. Hence, for any $x \in S$, $y \in S$, we must have $V(x) \cdot V(y) \neq 0$. Choose $a \in V(x) \cdot V(y)$. Since $V(x) \subset V(x)$, $V(y) \subset V(y)$, we have $V(a) \subset V(x) \cdot V(y)$. Since for any set $V(a)$ we have $V(a) \cdot V(z) \neq 0$, it follows that $V(a) \cdot V(x) \cdot V(y) \neq 0$. By a simple induction it follows that every finite collection of the sets $\{ V(x), x \in S \}$ has a nonempty intersection. Hence, by the usual compactness argument, we have $\prod_{x \in S} V(x) \neq 0$, if $S$ is not convex.

Finally, to prove $\prod_{x \in S} V(x)$ is connected if $S$ is not convex, we first prove $V(x) \cdot V(y)$ is connected. Suppose this were not so, and let $K_1$ and $K_2$ be two components of $V(x) \cdot V(y)$. Since $K_1$ and $K_2$ are each connected closed sets of visibility in $S$, each contains a minimal closed connected set of visibility. Hence, as proved above, we must have $K_1 \cdot K_2 \neq 0$ if $S$ is not convex. The fact that $\prod_{x \in S} V(x)$ is connected follows by a simple induction together with the fact that if every finite subcollection of a collection of continua have a connected intersection, then they all have a connected intersection. This completes the proof of Theorem 1.

2. Minimal compact convex sets of visibility. In this section we confine ourselves to sets $S \subset E_2$.

Lemma 1. Let $S$ be a compact set in $E_2$. Suppose each point $x \in S$ is contained in a unique minimal closed convex set of visibility $V(x)$ in $S$. Then $S$ is simply connected.$^3$

Proof. Suppose $S$ is not simply-connected, and let $K$ be a bounded component of the complement of $S$. Let $H(K)$ be the convex hull of $K$, where $\overline{K}$ is the closure of $K$. Let $B(H)$ denote the boundary of $H(K)$. There exists a point $x \in B(H)$ such that a unique line of support $L$ to $H(K)$ at $x$ exists. If $x \in \overline{K}$, let $x = y$. If $x \notin \overline{K}$, let $L_1$ be the line through $x$ perpendicular to $L$, and let $y$ be the point of $L_1 \cdot \overline{K}$ which is nearest to $x$. Since $H(K)$ is bounded, there exists a unique line of support $L^*$ to $H(K)$ which is parallel to $L$, and distinct from $L$. Clearly since $K$ is an open connected set, $y \cdot L^* = 0$. Let $L^* \cdot \overline{K} = G$.

$^3$ A set in $E_4$ is simply-connected if each component of its complement is unbounded.
To prove that \( G \) is a single point, suppose there exist two points \( u \in G, v \in G \). Let \( a \) be any point between \( u \) and \( v \) on \( L^* \), and let \( L(a) \) be the line through a perpendicular to \( L^* \). Let \( b \) be the point of \( K \cdot L(a) \) which is nearest to \( a \). The line segment of \( S \) which joins \( b \) to a point of \( V(y) \) and the segment \( ab \) (degenerate or not) violates the connectedness of \( K \), since \( u \) and \( v \) are limit points of \( K \). Hence, \( L^* \cdot K = L^* \cdot B(H) = p \), a point of \( S \). Moreover, the line \( L^* \) is not a unique line of support to \( H(K) \) at \( p \), otherwise \( V(y) \) would not be visible from \( p \).

Now, let \( L_i \) be a sequence of parallel lines between \( L \) and \( L^* \) such that \( L_i \rightarrow L^* \) as \( i \rightarrow \infty \). Choose \( r_i \in L_i \cdot B(K) \), \( s_i \in L_i \cdot B(K) \) such that the segment \( r_i s_i \) contains the set \( L_i \cdot K \), and such that in terms of a direction on \( L, L^* \), and \( L_i \) we have \( r_i < s_i \) on \( L_i \). Due to the position of the point \( y \), defined above, the visibility of \( V(r_i) \) and \( V(s_i) \) implies that \( V(r_i) \) and \( V(s_i) \) must intersect \( L \) on opposite sides of \( x \) relative to \( L \). In fact, \( V(r_i) \cdot L \) and \( V(s_i) \cdot L \) have the same order on \( L \) as \( r_i \) and \( s_i \) have on \( L_i \). Since \( L^* \cdot B(H) = p \in S \), it follows that \( r_i \rightarrow p, s_i \rightarrow p \) as \( i \rightarrow \infty \). Each of the collections \( \{ V(r_i) \} \) and \( \{ V(s_i) \} \) contains a convergent subsequence which converges to a closed convex set of visibility \( V_r \) and \( V_s \) respectively, with \( p \in V_r, p \in V_s \). Let \( R_+ \) be the closed half-plane bounded by \( L^* \) which does not contain the point \( x \). Since \( V_r \cdot L \neq 0, V_s \cdot L \neq 0 \), with \( x \) between \( V_r \cdot L \) and \( V_s \cdot L \), and since \( p \in B(K) \), it follows that \( V_r \cdot V_s \subset R_+ \). On account of the uniqueness of \( V(p) \), we have \( V(p) \subset V_r, V(p) \subset V_s \). Hence, \( V(p) \subset V_r \cdot V_s \subset R_+ \). However, due to the position of the point \( y \), there exists no point \( q \in V(p) \) such that \( y q \in S \) (\( K \) is an open connected set). This is a contradiction; hence, \( S \) is simply connected.

**Lemma 2.** Assume the same hypotheses about \( S \) as in Lemma 1. Suppose there exists two points \( x \) and \( y \) in \( S \) such that \( V(x) \cdot V(y) = 0 \). Then \( S \) is starlike.

**Proof.** A line \( L \) divides the plane into two closed half-planes, denoted by \( R_+ \) and \( R_- \). A mutually separating line of support to \( V(x) \) and \( V(y) \) is one which is a line of support to each, and one for which either

\[
V(x) \subset R_+, \quad V(y) \subset R_- \quad \text{or} \quad V(x) \subset R_-, \quad V(y) \subset R_+.
\]

If \( V(x) \) and \( V(y) \) are not collinear, there exist two mutually separating lines of support to \( V(x) \) and \( V(y) \), denoted by \( L_1 \) and \( L_2 \). If \( V(x) \) and \( V(y) \) are collinear, then \( L_1 = L_2 \). If \( L_1 \neq L_2 \), let \( p = L_1 \cdot L_2 \). If \( L_1 = L_2 \), choose \( p \in L_1 \) between \( x \) and \( y \), with \( p \in V(x), p \in V(y) \). Let \( r_i \in L_i \cdot V(x), s_i \in L_i \cdot V(y) \) \((i = 1, 2)\). Since \( V(y) \) is a minimal set of visibility,
there exist points \( p_1 \in V(y) \), \( p_2 \in V(y) \) such that \( r_1 p_1 \subset S \), \( r_2 p_2 \subset S \). The quadrilateral \( r_1 p_1 p_2 r_2 \) (degenerate or nondegenerate) may be simple or not, but in any case its sides all belong to \( S \). Since \( L_1 \) and \( L_2 \) are mutually separating lines of support to \( V(x) \) and \( V(y) \), it is easily seen that triangle \( r_1 r_2 p \subset r_1 p_1 p_2 r_2 \). Since, by Lemma 1, \( S \) is simply-connected, we must have triangle \( r_1 r_2 p \subset S \). Hence the convex hull \( H[p + V(x)] \subset S \). In exactly the same manner, we have \( H[p + V(y)] \subset S \). Since \( V(p) \subset H[p + V(x)] \), \( V(p) \subset H[p + V(y)] \), and since \( H[p + V(x)] \cdot H[p + V(y)] = p \), the uniqueness of \( V(p) \) implies \( V(p) = p \), so that \( S \) is starlike.

The following definition is due to Brunn [1].

**Definition 3.** The set \( K(S) = \{ x \in S, V(x) = x \} \) is called the Kerneigebiet of \( S \). (The set \( S \) is starlike relative to each point of the Kerneigebiet.)

**Theorem 2.** Let \( S \) be a compact set in \( E_2 \), and suppose each point \( x \in S \) is contained in a unique minimal closed convex set of visibility \( V(x) \) in \( S \). Then either \( S \) is convex or \( S \) is starlike with respect to one and only one point of \( S \). (In other words, the Kerneigebiet \( K(S) \) is either \( S \) or it is a single point of \( S \).)

**Proof.** Suppose \( S \) is not starlike. Then by Lemma 2, for each pair of points \( x \in S \), \( y \in S \) we have \( V(x) \cdot V(y) \neq 0 \). Then by exactly the same argument as given in Theorem 1, involving the finite intersection property and compactness, we must have \( \prod x \in S V(x) \neq 0 \). But this is a contradiction, since \( \prod x \in S V(x) \subset K(S) \). Hence, \( S \) is starlike. Suppose there exist two distinct points \( a \in K(S) \), \( b \in K(S) \). If \( S \subset L(a, b) \), then \( S \) is a line segment. If \( S \subset S - L(a, b) \), then \( V(x) \subset a \), \( V(x) \subset b \). However, this implies \( V(x) = z \) so that \( z \in K(S) \). Similarly, if \( c \in L(a, b) - a - b \). Then \( V(c) \subset a \), \( V(c) \subset b \). Thus, \( K(S) = S \) if \( a \neq b \). Thus, either \( K(S) = S \) or \( K(S) \) is a single point of \( S \). This completes the proof of Theorem 2.

There exist a variety of interesting examples of the set \( S \) in Theorem 2. For instance, the set consisting of two externally tangent circular disks is a nonconvex one containing interior points.

The corresponding theory for unbounded closed sets \( S \subset E_2 \) offers considerably more difficulty. Although I am able to establish a nontrivial generalization of Theorem 2 when at least one of the sets \( V(x) \) is bounded, the case when all the \( V(x) \) are unbounded remains unsettled.

**3. Concluding remarks.** In a previous paper [3] Straus and Valentine proved the following theorem.
“Let $S$ be a closed connected set in a finite dimensional linear space, and let $R_n$ be the subspace of minimal dimension which contains $S$. Then the set $S$ is convex if and only if each point $x \in S$ is contained in a unique maximal convex subset of $S$ of dimension greater than or equal to $n-1$.”

Observe that the notion of visibility is not required in the above uniqueness requirement. This cannot be done for minimal convex sets of visibility since a minimal convex set of $S$ containing a point $x$ is always $x$. This is the reason the theory in this paper differs essentially from that used by Straus and Valentine.

The generalization of Theorem 2 to $E_n$ ($n > 2$) remains unsettled, and it appears to offer considerable difficulties. Finally, the converse of Theorem 2 is clearly false. For instance, a circular disk together with two outward normals (segments) is an obvious counterexample.

Bibliography


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