ON AN OPEN QUESTION CONCERNING FIXED POINTS
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A space $X$ is said to have the f.p.p. (fixed point property) if every continuous function $f$ from $X$ to $X$ has a fixed point. Whether if $X$ and $Y$ have the f.p.p. then $X \times Y$ has the f.p.p. is an open question.

A space $X$ is said to have the F.p.p. (fixed point property for multi-valued functions) if every continuous multi-valued function $F$ from $X$ to $X$ has a fixed point, i.e., a point $x$ such that $x \in F(x)$. Interest in fixed points for multi-valued functions leads one to question under what conditions on the spaces $X$ and $Y$ and on the multi-valued function $F$ on $X$ to $Y$ there will exist a continuous trace $f$ of $F$, that is, a continuous function $f$ on $X$ to $Y$ such that $f(x) \in F(x)$ for all $x$. For some specific multi-valued functions $F$ it is possible to produce a continuous trace. It is by use of these traces that most of the fixed point theorems in the literature for multi-valued functions are proved. In fact the open question mentioned above (which is concerned only with single-valued functions) can be answered if one can produce continuous traces of two particular multi-valued functions. This paper proves some fixed point theorems by producing continuous traces, shows that a continuous multi-valued function need not have a continuous trace, and gives an example which indicates that a general theorem on the existence of a continuous trace is not likely to be established without strong conditions on $F$ regardless of what conditions are placed on $X$ and $Y$. This example answers in the negative the generalization of the above open question to the multi-valued case, exhibits a continuous multi-valued function which has no continuous trace, shows that the general Tychonoff cube does not have the F.p.p., and shows that a space with the f.p.p. need not have the F.p.p.

**Notation.** By $\{x_a\} \to x_0$ we denote a sequence of points indexed by a directed set $A$ and converging to $x_0$. The directing relation in $A$ will be denoted by $\cdot$.

**Definition 1.** *Continuous.* A multi-valued function on a space $X$ to a space $Y$ is said to be continuous at $x_0$ if $\{x_a\} \to x_0$ implies that $F(x_0) = \text{cofinal limit} \{F(x_a)\} = \text{residual limit} \{F(x_a)\}$. $F$ is said to be

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continuous if it is continuous at every \( x \) in \( X \).

For the details of how this definition of continuity is related to the definitions used elsewhere in the literature see [1]. This definition is strong enough, however, to insure that the example is valid for functions continuous under the definitions used by Ratner [2], Wallace [3], Eilenberg and Montgomery [4], Kakutani [5], Banach and Mazur [6], and Michael [7].

Directly from the definition of trace we establish the following

**Lemma 1.** Let \( f \) be a trace of a multi-valued function \( F \) on \( X \) to \( Y \) and let \( x \) be a fixed point of \( f \). Then \( x \) is a fixed point of \( F \).

From Lemma 1 it is clear that a sufficient condition for a continuous multi-valued function \( F \) to have a fixed point is that \( F \) have a continuous trace which has a fixed point. But this is not a necessary condition. In fact one obtains a continuous multi-valued function with a fixed point and no continuous trace by defining \( F \) from the unit circle at the origin in the complex plane to itself by \( F(z) = \) the two square roots of \( z \).

The example.

**Theorem 1.** A bounded closed interval \( I \) of real numbers has the F.p.p.

**Proof.** Brouwer's theorem assures us that \( I \) has the f.p.p. Hence in view of Lemma 1 it is sufficient to prove that every continuous function \( \hat{F} \) on \( I \) to \( I \) has a continuous trace. We shall in fact prove that if \( R \) is a bounded closed interval of real numbers and \( F \) is a continuous multi-valued function on a space \( X \) to \( R \), then \( F \) has a continuous trace \( f \). Define \( f \) on \( X \) to \( R \) by \( f(x) = \text{lub} \{ y | y \in F(x) \} \).

It is known [1] that a multi-valued function from a space \( X \) to a compact Hausdorff space \( Y \) is continuous if and only if \( x_0 \in X \) implies:

1. \( F(x_0) \) is closed,
2. \( V \) open containing \( F(x_0) \) implies that there exists an open set \( U' \) containing \( x_0 \) and such that whenever \( x \in U' \) then \( F(x) \subseteq V \), and
3. \( y_0 \in F(x_0) \), \( y_0 \in V \), and \( V \) open imply that there is an open set \( U'' \) containing \( x_0 \) such that whenever \( x \in U'' \) then \( F(x) \cap V \neq \emptyset \).

Let \( V_{\phi} \) be an open interval of length \( 2\phi \) with center \( f(x') \), where \( \phi \) is a positive real number. Then \( V_{\phi} \) is also an open set containing \( f(x') \). By (3) there is an open set \( U'' \) containing \( x_0 \) such that \( x \in U'' \) implies that \( F(x) \cap V_{\phi} \neq \emptyset \). Hence \( x \in U'' \) implies that \( \text{lub} \{ y | y \in F(x) \} = f(x) \geq f(x') - \phi \). Let \( V = \{ y | y < \phi + f(x') \} \). This set \( V \) is open containing \( F(x') \). Hence (2) implies that there exists an open set \( U' \) containing \( x' \) such
that \( x \in U' \) implies that \( F(x) \subseteq V \). Then \( x \in U' \) implies \( f(x) = \text{lub} \{ y \mid y \in F(x) \} \leq f(x') + \phi \). Let \( U = U' \cap U'' \). Then \( x \in U \) implies that \( |f(x) - f(x')| \leq \phi \), therefore \( f(x) \subseteq V_{2\phi} \), and \( f \) is continuous at \( x' \).

Theorem 1 established that \( I \) has the F.p.p. Let \( r \) be a continuous function which retracts the unit square \( I \times I \) onto the unit disc \( X \). Define \( F \) from \( X \) to \( X \) as follows. If \( x \) is the origin, let \( F(x) = S \), where \( S \) denotes the unit circle with center at the origin. If \( x \) is not the origin: (a) Extend the segment from the origin through \( x \) until it meets \( S \) in a point \( A \). (b) Draw a perpendicular at \( x \) to the radius constructed in (a) and denote its intersections with \( S \) as \( B \) and \( C \). (c) Consider the closed arc \( BA \) on \( S \). Let \( MBA \) be the closed arc of \( S \) with center \( A \), length twice the length of the arc \( BA \), and having end points \( M \) and \( N \). (d) Let \( F(x) = MBA \). That \( F \) has the three properties utilized in the proof of Theorem 1 is geometrically evident and hence \( F \) is continuous. Define \( G \) to be \( iF \) followed by a rotation of ninety degrees, where \( i \) denotes the injection of \( X \) into \( I \times I \). The continuity of \( G \) follows from \([1, \text{Proposition 18}]\). Now \( x \in (I \times I) - S \) implies that \( x \in G(x) \) because \( G(I \times I) \subseteq S \). Also \( x \in S \) implies that \( F(x) = x \) and the rotation moves \( x \). Hence \( G \) has no fixed point and consequently \( I \times I \) does not have the F.p.p.

**Cartesian and apex functions.** Most theorems on fixed points for multi-valued functions demand either that \( F(x) \) be a connected set for every \( x \) or that \( F(x) \) be a convex set for every \( x \). The literature appears to be void of fixed point theorems with no condition on the image of a point. Theorem 1 above is such a theorem. The example above shows that the two-dimensional cube does not have the F.p.p. A simple extension of this result shows that no Tychonoff cube of dimension greater than one has the F.p.p. Then in order to prove theorems concerning fixed points for functions on a Tychonoff cube one must place some further conditions on \( F \). The following theorems indicate that \( F \) may enjoy much greater pointwise freedom than is allowed under the usual assumption that \( F(x) \) is either convex or connected.

**Definition 2. Cartesian function.** Let \( T = PI \) be a Tychonoff cube. A subset \( Y_0 \) of \( T \) is called a \( T \)-cartesian subset if \( Y_0 = PM_0 \), where \( M_0 \) is a subset of \( I_0 \). A function \( F \) from a space \( X \) to a space \( Y \) is called a cartesian function if there exists a homeomorphism \( h \) of \( Y \) into some Tychonoff cube \( T \) such that (1) \( h(Y) \) is a retract of \( T \) and (2) \( x \in X \) implies that \( hF(x) \) is a \( T \)-cartesian subset of \( T \).

**Theorem 2.** Every continuous Cartesian function from a space \( X \) to itself has a fixed point.
Proof. Let $F$ be a continuous cartesian function on $X$ to itself. There is a Tychonoff cube $T = P I_a$, a homeomorphism $h$ of $X$ into $T$, and a retraction $r$ of $T$ onto $h(T)$ such that $x \in X$ implies that $hF(x)$ is a $T$-cartesian subset of $T$. Let $G(x) = ihF(x)$, where $i$ denotes the injection of $h(X)$ into $T$. Define $H$ on $T$ to $T$ by $H(x) = Gh^{-1}r(x) = ihFh^{-1}r(x)$ and define $H_a$ from $T$ to $I_a$ by $H_a(x) =$ the projection of $H(x)$ in $I_a$. That $H$ and $H_a$ are continuous is established in [1]. In the proof of Theorem 1 it was shown that $H_a$ has a continuous trace $f_a$. Kakutani [5] showed that if each $f_a$ is a continuous single-valued function, so is $f$ defined by $f(x) = Pf_a(x)$. For each $a$, $f_a(x)$ is an element of $H_a(x)$ and $H(x) = PM_a(x)$, where $M_a$ is a subset of $I_a$, so that $f(x) \in H(x)$. Now $f$ is a continuous single-valued function from a Tychonoff cube to itself and hence has a fixed point.

Let $x'$ be a fixed point of $H$. Then $x' \in H(x') = ihFh^{-1}r(x') = hFh^{-1}r(x')$. Since $x' \in T$, $r(x') \in h(T)$ and hence there exists $x'' \in T$ such that $r(x') = h(x'')$. Then $x' \in H(x') = hFh^{-1}h(x'') = hF(x'') \subseteq hF(X) \subseteq h(X)$. The function $r$ retracts $T$ onto $h(X)$, hence $x' = r(x') = h(x'') = h^{-1}(x')$, $x' \in hF(x')$, and $x'' = r^{-1}(x') \subseteq F(x'')$.

Definition 3. Apex set. Let $B$ be a closed subset of a Tychonoff cube $T = PI_a$. Denote by $B_a$ and $b_a$ the projections of $B$ and of $b$, respectively, in $I_a$. For a fixed $a$, denote $\{b_a \mid b_a \subseteq B_a\}$ by $m(B_a)$. The set $B_a$ is closed and hence $m(B_a) \subseteq B_a$. If there is only one point in $B$ which projects onto $m(B_a)$ we say that $B$ is an apex subset of $T$ with respect to $a$.

Definition 4. Apex function. A function $F$ from a space $X$ to a space $Y$ is called an apex function if there is a homeomorphism $h$ of $Y$ onto a retract of a Tychonoff cube $T = PI_a$ such that, for some fixed $a = a(1)$, $x \in X$ implies that $hF(x)$ is an apex subset of $T$ with respect to $a(1)$.

Theorem 3. Every continuous apex function from a space $X$ to itself has a fixed point.

Proof. Let $G$ be a continuous apex function from $X$ to $X$. Then there is a Tychonoff cube $T = PI_a$, a homeomorphism $h$ of $X$ into $T$, a retraction $r$ of $T$ onto $h(X)$, and a fixed $a = a(1)$ such that $x \in X$ implies that $hG(x)$ is an apex subset of $T$ with respect to $a(1)$. The function $F = hGh^{-1}r$ is defined on $T$ to $T$. As in the proof of Theorem 2, $F_{a(1)}$ is continuous. By (1) in the proof of Theorem 1, $t \in T$ implies that $F_{a(1)}(t)$ is closed and hence $t \in T$ implies that $m[F_{a(1)}(t)]$ is an element of $F_{a(1)}(t)$. The hypothesis that $G$ is an apex function implies that there is exactly one $t' \in F(t)$ such that $t'$ projects onto $m[F_{a(1)}(t)]$. Define a single-valued function $f$ from $T$ to $T$ by $f(t) = t'$. 
Clearly \( f \) is a trace of \( F \).

Let \( x^0 \) be an element of \( T \) and let \( y^0 \) be an element of \( F(x^0) \). Let \( \phi \) be a real number greater than zero and let \( V_{\alpha(t)}(\phi) \) contain \( f_{\alpha(t)}(x^0) = y^0_{\alpha(t)} \). By (3) in the proof of Theorem 1 there is an open set \( U'' \) containing \( x^0 \) such that whenever \( x \in U'' \), then \( F_{\alpha(t)}(x) \cap V_{\alpha(t)}(\phi \neq 0) \), and hence \( f_{\alpha(t)}(x) = \operatorname{lub} \{ y_{\alpha(t)} \mid y_{\alpha(t)} \in F_{\alpha(t)}(x) \} \) is greater than \( y_{\alpha(t)} - \phi \).

Let \( W = \{ y_{\alpha(t)}, y_{\alpha(t)} < y^0_{\alpha(t)} + \phi \}. \) Then \( W \) is open and contains \( F_{\alpha(t)}(x^0) \) so that (2) in the proof of Theorem 1 implies the existence of an open set \( U' \) containing \( x^0 \) such that \( F(x) \) is contained in \( W \) for all \( x \in U' \), i.e., \( x \in U' \) implies \( \operatorname{lub} \{ y_{\alpha(t)} \mid y_{\alpha(t)} \in F_{\alpha(t)}(x) \} < y^0_{\alpha(t)} + \phi \).

Now \( x \in U' \cap U'' \) implies that \( y_{\alpha(t)} - \phi < y_{\alpha(t)} = f_{\alpha(t)}(x) < y^0_{\alpha(t)} + \phi \), and hence \( f_{\alpha(t)} \) is continuous.

Let \( f(x^0) = y^0 \). Assume that \( f \) is not continuous at \( x^0 \). Then there exists an open set \( W \) containing \( y^0 \), a directed set \( D \), and a sequence \( \{ x_d \} \rightarrow x^0 \) such that \( d \in D \) implies that \( f(x_d) \in W \). The set \( T - W \) is closed and hence compact. The sequence \( \{ f(x_d) \} \) determines a net \( \phi \) on \( D \) to \( T - W \) defined by \( \phi(d) = f(x_d) \) and hence [9, Theorem 24] there exists a subnet \( (E, \theta) \) with a limit \( y \) in \( T - W \). Let \( k \) be the function on \( E \) to \( D \) satisfying the definition of subnet. Since \( \phi(d) = f(x_d) \), \( \theta(e) = \phi(k(e)) = f(x_{k(e)}) \). If \( V \) is an open set containing \( y \), then there exists \( e(V) \in E \) such that \( e \cdot e(V) \) implies that \( \theta(e) \in V \). If \( U \) is an open set containing \( x^0 \), then there exists \( d' \) such that \( d \ast d' \) implies that \( x_d \in U \). If \( d' \in D \) then there exists \( e' \) such that \( e \ast e' \) implies that \( k(e) \ast d' \). Therefore \( e \ast e' \) implies that \( x_{k(e)} \in U \). Then \( \{ x_{k(e)} \} \rightarrow x^0 \) and \( \{ f_{\alpha(t)}(x_{k(e)}) \} \rightarrow y_{\alpha(t)} \). It was shown that \( f_{\alpha(t)} \) is continuous, hence \( f_{\alpha(t)}(x^0) = y^0_{\alpha(t)} \). But \( f_{\alpha(t)}(x^0) = y^0_{\alpha(t)} \), therefore \( y_{\alpha(t)} = y^0_{\alpha(t)} \). Then the hypothesis that \( F \) is an apex function implies that \( y = y^0 \), which is an element of \( W \). But \( y \in T - W \). This contradiction implies that our assumption was false and \( f \) is continuous. This trace \( f \) is a continuous single-valued function on a Tychonoff cube to itself and hence Lemma 1 implies that \( F \) has a fixed point \( x_0 \). The proof that \( h^{-1}(x_0) \) is a fixed point of \( G \) is a reiteration of the last statement in the proof of Theorem 2.

Bibliography

2. L. Ratner, Multi-valued transformations, University of California, 1949.

Errata, Volume 3

C. W. Curtis, A note on noncommutative polynomials.
p. 965, line 10 from the bottom. Add to condition (b): “where \( T(r) \neq 0 \) if \( r \neq 0 \).”

Errata, Volume 4

W. R. Mann, Mean value methods in iteration.
p. 507, Display (2) should include the following:

\[
\lim_{t \to \infty} a_{ij} = 0 \quad \text{for all } j.
\]

E. Michael, A note on paracompact spaces.
p. 835, diagram near the top of the page. For “covering” read “open covering” (twice), and for “refinement” read “open refinement” (twice).