A NOTE ON PREHARMONIC FUNCTIONS
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1. Let $L$ be the set of points whose coordinates are rational integers. Let $D$ be a domain, that is to say, an open connected set, and let $G$ be the set $D \cdot L$. A point $P(m, n)$ of $G$ is an interior point if the four points $(m \pm 1, n), (m, n \pm 1)$ contiguous to $P$ belong to $G$. Otherwise $P$ is a boundary point.

A function $f(m, n)$ defined on $G$ is preharmonic if the value of $f$ at any interior point is the mean of the values of $f$ at the contiguous points, that is to say

$$4f(m, n) = f(m + 1, n) + f(m - 1, n) + f(m, n + 1) + f(m, n - 1).$$

For several decades the subject of preharmonic functions has been considered by many mathematicians, and the connection with harmonic functions has long been known. A recent paper by Heilbronn [1] states a number of theorems which are the analogues of classical theorems for harmonic functions.

In this note we consider functions which are preharmonic and non-negative in the half-plane $m \geq 0$ and prove a representation theorem analogous to that for positive harmonic functions [2], and a theorem which is the analogue of the Phragmén-Lindelöf type theorem for positive harmonic functions [3; 4].

2. We require the following lemmas:

**Lemma 1.** If $f(m, n)$ is preharmonic on a bounded domain $D$, then $f(m, n)$ is either constant or attains its maximum and minimum on $D$ on the boundary only.

**Lemma 2.** If $f(m, n)$ is preharmonic everywhere and satisfies the inequality

$$|f(m, n)| < A \{1 + (|m| + |n|)^k\}$$

for all $m, n$, where $k$ is a positive integer, then $f(m, n)$ is a polynomial of degree not exceeding $k$.

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1 In what follows, $A$ will always represent a positive nonzero number, independent of the variables in the context.
These lemmas are special cases of Theorems 1 and 6 of Heilbronn's paper.

**Lemma 3.** The function

\[ h(m, n) = \frac{1}{\pi} \int_0^\pi \cos mt \phi^n(t) dt, \]

where \( \phi(t) \) is the smaller root of the equation

\[ \phi(t) + \phi^{-1}(t) + 2 \cos t = 4, \]

is preharmonic everywhere with the following properties:

(a) \( h(0, 0) = 1 \),

(b) \( h(m, 0) = 0 \) for \( m \neq 0 \),

(c) \( h(m, n) > 0 \) for \( n > 0 \),

(d) \( \left| h(m, n) - n/\pi(m^2 + n^2) \right| \leq A/n(m^2 + n^2) \) for all \( m \) and positive \( n \).

(a) and (b) follow by inspection. To prove (c), let

\[ M(n) = \operatorname{glb}_{0 < \phi < \pi} h(m, n) \]

for \( n \geq 0 \). It is easily seen that \( |\phi(t)| < 1 \) for \( 0 < t \leq \pi \) and so \( M(n) \to 0 \) as \( n \to \infty \) and, from the difference equation for preharmonic functions, we have for \( n \geq 1 \)

\[ 2M(n) \geq M(n + 1) + M(n - 1) \]

and, since \( M(0) = 0 \), the result follows.

It may be verified that \( \phi(t) \) is a positive decreasing function of \( t \) in \( (0, \pi) \) with derivatives of all orders there, that

(1) \[ \phi'(\pi) = 0, \quad \lim_{t \to 0^+} \phi'(t) = -1, \]

(2) \[ \phi(t) = 1 - t + t^2/2 - t^4/12 + O(t^6) \]

as \( t \to 0^+ \), and that there exists a real number \( \eta > 0 \) such that

(3) \[ \phi(t) \leq e^{-\eta t} \quad \text{for } 0 \leq t \leq \pi. \]

Integrating by parts twice in the expression for \( h(m, n) \) we have from (1) and the fact that \( \sin m\pi = 0 \)

\[ \pi h(m, n) = \frac{n}{m^2} - \frac{n}{m^2} \int_0^\pi \phi^{n-2}(t) \cos mt \left( (n - 1) \left[ \phi'(t) \right]^2 + \phi(t)\phi''(t) \right) dt, \]

or, adding \( \pi(n^2/m^2)h(m, n) \) to each side,
\[ \pi \frac{m^2 + n^2}{m^2} \cdot h(m, n) = \frac{n}{m^2} - \frac{n}{m^2} \int_0^\pi \phi^{n-2}(t) \psi(t) \cos mtdt, \]

where

\[ \psi(t) = (n - 1) \left( \phi'(t) \right)^n + n \phi(t) \phi''(t) - n \phi^2(t). \]

From the enunciated properties of \( \phi(t) \) we may easily show that

\[ |\psi(t)| < A(n \mid t^2 + t) \]

for \( 0 < t \leq \pi \). Thus, by (3), we have for \( n \geq 1 \)

\[ \left| \pi \frac{m^2 + n^2}{m^2} \cdot h(m, n) - \frac{n}{m^2} \right| < A \frac{n}{m^2} \int_0^\pi e^{-\pi t} (nt^2 + t) dt \]

\[ < \frac{A}{nm^2}, \]

and this completes the proof of Lemma 3.

3. Theorem 1. A necessary and sufficient condition for a function \( f(m, n) \) to be non-negative and preharmonic for \( n \geq 0 \) is that the numbers \( f(m, 0) \{ m = 0, \pm 1, \pm 2, \ldots \} \) should be non-negative and satisfy

\[ \sum_{m=-\infty}^{\infty} \frac{f(m, 0)}{1 + m^2} < \infty \]

and that there should exist a non-negative number \( D \) for which

\[ f(m, n) = Dn + \sum_{r=-\infty}^{\infty} f(r, 0) h(m - r, n) \]

for \( n \geq 0 \).

Sufficiency. For any large positive \( N \) and \( n > 0 \) we have, from Lemma 3(d),

\[ \sum_{r=-N}^{N} f(r, 0) h(m - r, n) < A \sum_{r=-N}^{N} \frac{f(r, 0)n}{(m - r)^2 + n^2} \]

\[ < AC(m, n) \sum_{r=-N}^{N} \frac{f(r, 0)}{1 + r^2} \]

where

\[ C(m, n) = \text{lub} \frac{n(1 + r^2)}{(m - r)^2 + n^2} \]

and is finite for any fixed \( m, n \). Thus the function defined by (4) is
an absolutely convergent series of non-negative preharmonic functions and, hence, is itself non-negative and preharmonic for \( n \geq 0 \).

**Necessity.** Let \( R \) be a positive integer and define

\[
 f_R(m, n) = f(m, n) - \sum_{r=-R}^{R} f(r, 0) h(m - r, n).
\]

Evidently \( f_R(m, n) \) is preharmonic in the half-plane \( n \geq 0 \) and also

\[
 f_R(m, n) \geq - \left\{ \max_{|r| \leq R} h(m - r, n) \right\} \sum_{r=-R}^{R} f(r, 0).
\]

From Lemma 3(d) the right-hand side has arbitrarily small modulus for all points \((m, n)\) of the half-plane lying outside a sufficiently large circle with centre at \((0, 0)\). Since, by Lemma 1, a preharmonic function in a finite domain attains its minimum on the boundary and \( f_R(m, n) = 0 \) for \( n = 0 \), it follows that for \( n \geq 0 \), \( f_R(m, n) \geq 0 \). That is to say, for \( n \geq 0 \) we have

\[
 f(m, n) \geq \sum_{r=-R}^{R} f(r, 0) h(m - r, n),
\]

and letting \( R \to \infty \)

\[
 (5) \quad f(m, n) \geq \sum_{r=-\infty}^{\infty} f(r, 0) h(m - r, n).
\]

Next, by Lemma 3(d), there exists a large positive integer \( N \) for which

\[
 h(m, N) > 1/(N^2 + m^2)
\]

for all integers \( m \). Thus we have, from \( 5 \),

\[
 f(0, N) \geq \sum_{r=-\infty}^{\infty} f(r, 0) h(-r, N)
\]

\[
 \geq \sum_{r=-\infty}^{\infty} f(r, 0) \frac{1}{r^2 + N^2}
\]

\[
 = \frac{1}{N^2} \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2}.
\]

This proves that if \( f(m, n) \) is non-negative and preharmonic for \( n \geq 0 \),

\[
 \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2} < \infty.
\]

If we write
it remains to show that \( f_\infty(m, n) = Dn \) for some non-negative \( D \). Now since the series \( \sum_{\infty} f(r, 0) h(m - r, n) \) is convergent and each term is non-negative and preharmonic for \( n \geq 0 \), \( f_\infty(m, n) \) also is non-negative and preharmonic for \( n \geq 0 \), and, a fortiori, for any integral \( t > 0 \), \( f_\infty(m, n + t) \) is non-negative and preharmonic for \( n \geq 0 \). From what we have just proved above, we have

\[
\sum_{r=-\infty}^{\infty} \frac{f_\infty(r, t)}{1 + r^2} < \infty
\]

and, a fortiori, \( f_\infty(m, t) < K_t(1 + m^2) \), where \( K_t \) is finite for each integral \( t \). Let us assume for the moment that we have shown that

(6) \( f_\infty(m, n) < An^2(1 + m^2) < A [1 + (|m| + |n|)^4] \)

for \( n > 0 \). We may continue \( f_\infty(m, n) \) uniquely throughout the entire plane by writing

(7) \( f_\infty(m, -n) = -f_\infty(m, n) \)

for \( n > 0 \), and have

(i) \( f_\infty(m, n) \) preharmonic everywhere,

(ii) \( f_\infty(m, n) < A [1 + (|m| + |n|)^4] \) everywhere,

(iii) \( f_\infty(m, 0) = 0 \) for all \( m \),

(iv) \( \text{sign } f_\infty(m, n) = \text{sign } n \) for \( n \neq 0 \).

Applying Lemma 2 to \( f_\infty(m, n) \) it follows from (8)(ii) that \( f_\infty(m, n) \) is a polynomial of degree not exceeding 4. From (8)(iii), \( n \) must be a factor of \( f_\infty(m, n) \); since \( f_\infty(m, n) \) by (7) contains only odd powers of \( n \) we must have

\[
f_\infty(m, n) = n\phi(m, n^2),
\]

where \( \phi(m, n^2) \) is a polynomial of degree not exceeding 3. Further, from (8)(iv), \( \phi(m, n^2) \) is everywhere non-negative, and so of degree not exceeding 2. We have now shown that

\[
f_\infty(m, n) = n(\alpha m^2 + \beta n^2 + \gamma m + \delta)
\]

where \( \alpha \) and \( \beta \) are non-negative. It may be verified, from the difference equation, that since \( f_\infty(m, n) \) is preharmonic, then \( \alpha + 3\beta = 0 \). Thus \( \alpha = \beta = 0 \) and this implies that \( \gamma = 0 \) and \( \delta \geq 0 \). This completes the proof that
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for some non-negative $D$.

It remains to prove (6). Consider the function

$$g(m, n, m, 2\bar{n}) = \frac{1}{\bar{n}} \sum_{r=1}^{\infty} \sin \frac{r\pi(m - m + \bar{n})}{2\bar{n}} \cdot \frac{r\pi}{2} \sinh \alpha_r n$$

where $\alpha_r$ is the positive root of the equation

$$\cosh \alpha_r + \cos \frac{r\pi}{2\bar{n}} = 2.$$  

This function is preharmonic everywhere$^2$ and may be shown to satisfy

$$g(m, n, m, 2\bar{n}) = \begin{cases} 
0 & \text{for } m = m \pm \bar{n}, \\
0 & \text{for } n = 0, \\
0 & \text{for } 1 \leq |m - m| \leq \bar{n}, n = 2\bar{n}, \\
1 & \text{for } m = m, n = 2\bar{n}.
\end{cases}$$

Further,

$$g(\bar{m}, 1, \bar{m}, 2\bar{n}) = \frac{1}{\bar{n}} \sum_{r=1}^{\infty} \sin^2 \frac{r\pi}{2} \frac{\sinh \alpha_r}{2 \sinh 2\alpha_r \bar{n}}$$

$$= \frac{1}{\bar{n}} \frac{\sinh \alpha_1}{\sinh 2\alpha_1 \bar{n}}.$$  

From (9) we have

$$\cosh \alpha_r = 2 - \cos \frac{r\pi}{2\bar{n}} < \cosh \frac{r\pi}{2\bar{n}},$$

and so

$$\alpha_r < \frac{r\pi}{2\bar{n}},$$

and substituting this in the inequality for $g(\bar{m}, 1, \bar{m}, 2\bar{n})$ we deduce that

$$g(\bar{m}, 1, \bar{m}, 2\bar{n}) > A/\bar{n}^3.$$  

Let us suppose that (6) is not true. Then there exists an increasing sequence of integers $\{n_r\}$, and a corresponding sequence of integers $\{m_r\}$ such that as $r \to \infty$.

$^2$ This method of writing preharmonic functions as a sum of products is due to Phillips and Wiener [5].
We shall suppose first that the integers $n_*$ are even. Consider the function

$$f_\omega(m, n) = f_\omega(m, n) - f_\omega(m, n_*) g(m, n, m, n_*) .$$

From (10) and (11) it is apparent that $f_\omega(m, n) \geq 0$ on the boundary of the square $|m-m_*| \leq n_*, 0 \leq n \leq n_*$, and also, by Lemma 1, inside the square. In particular, we have

$$f_\omega(m, n_*) \geq f_\omega(m, n_*) g(m, n_*, 1, m_*, n_*) ,$$

and so, by (11),

$$f_\omega(m, n_*) < A n_* (1 + m_*^2) ,$$

which contradicts our assumption. Similarly, if the sequence is odd, we may show that

$$f_\omega(m, n_*) < A n_* (1 + m_*^2) .$$

**Corollary.** Suppose $f(m, n)$ to be preharmonic and non-negative in $n \geq 0$. Then, as $n \to \infty$ subject to the condition $am + bn = 0$ where $a$ and $b$ are integers,

$$f(m, n) - H(m, n) = D n + O \{ (m^2 + n^2)^{-1/2} \} ,$$

where $D$ is a non-negative number and

$$H(m, n) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} f(r, 0) \cdot \frac{n}{(m - r)^2 + n^2} .$$

This result follows immediately from Theorem 1 and Lemma 3(d).

4. If $H(re^{i\theta})$ is positive and harmonic in the half-plane $0 < \theta < \pi$, then it may be written as [2]

$$H(re^{i\theta}) = dr \sin \theta + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{r \sin \theta}{r^2 - 2rt \cos \theta + t^2} \cdot dg(t) ,$$

where $d$ is a non-negative number and $g(t)$ is a nondecreasing function such that

$$\int_{-\infty}^{\infty} \frac{dg(t)}{1 + t^2} < \infty .$$

**Lemma 4.** If $H(re^{i\theta})$ is defined by (12), $-1 < \rho < 1$, $0 < \phi < \pi$, $n$ is an integer and $\alpha$, $\delta$ are any positive numbers such that
\[ H(n\delta e^{i\theta}) \sim a(n\delta) \delta \]

as \( n \to \infty \), then

\[ H(re^{i\theta}) \sim ar^\theta \]

as \( r \to \infty \).

There is no loss in generality in assuming \( d = 0 \), and so as \( n \to \infty \)

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dg(t)}{(\delta^2 + 2\delta t \cos \phi + t^2)} \sim \alpha \csc \phi \cdot (n\delta)^{-1}.
\]

This is easily shown to imply that for \( x > 1 \)

\[
\frac{g(x) - g(-x)}{x^2} + \int_x^\infty \frac{d\{g(t) - g(-t)\}}{t^2} < A x^{p-1}.
\]

Further, it will be sufficient to prove that for \( |r - \sigma| \leq 1 \) and \( r \to \infty \)

\[ I(re^{i\theta}) - H(\sigma e^{i\theta}) = o(r^\theta). \]

Now from (12), since \( d = 0 \),

\[
\left| H(re^{i\theta}) - H(\sigma e^{i\theta}) \right| = \left| \frac{(r - \sigma) \sin \phi}{\pi} \int_{-\infty}^{\infty} \frac{(t^2 - \sigma^2)dg(t)}{(t^2 - 2rt \cos \phi + t^2)(\sigma^2 - 2\sigma t \cos \phi + t^2)} \right|
\]

\[
\leq A \left[ \frac{g(r) - g(-r)}{r^2} + \int_r^\infty \frac{d\{g(t) - g(-t)\}}{t^2} \right]
\]

\[
\leq Ar^{p-1} = o(r^\theta)
\]

as \( r \to \infty \).

The following two lemmas are contained in a paper by Allen and Kerr [4].

**Lemma 5.** If \( H(re^{i\theta}) \) is defined by (12), \(-1 < \rho < 1\), and

\[ H(re^{i\theta/2}) \sim (1 + \rho)ar^\rho \sec \rho \phi/2 \]

as \( r \to \infty \), then

\[ g(x) \sim g(-x) \sim 2\alpha x^{\rho}\]

as \( x \to \infty \).

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*Actually Allen and Kerr state their results for the case \( r \to 0^+ \), but the case \( r \to \infty \) is an elementary corollary.*
Lemma 6. If $H(re^{i\theta})$ is defined by (12), $-1 < \rho < 1$, and

\begin{equation}
H(re^{i\theta}) \sim (1 + \rho) \csc \rho \pi \left[ \alpha \sin \rho (\pi - \theta) + \beta \sin \rho \theta \right] r^\rho,
\end{equation}

as $r \to \infty$ for two distinct values of $\theta$, then (13) remains true for all values of $\theta$ and as $x \to \infty$

\begin{equation}
g(x) - g(0) \sim \alpha x^{1+\rho}, \quad g(0) - g(-x) \sim \beta x^{1+\rho}.
\end{equation}

Theorem 2. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, $-1 < \rho < 1$, and

\begin{equation}
f(0, n) \sim (1 + \rho) \alpha \sec \rho \pi/2 \cdot n^\rho,
\end{equation}

as $n \to \infty$, then

\[
\sum_{m=-R}^{R} f(m, 0) \sim 2\alpha R^{1+\rho},
\]

as $R \to \infty$.

Theorem 3. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, $-1 < \rho < 1$, and

\begin{equation}
f(m, n) \sim (1 + \rho) \csc \rho \pi \left[ \alpha \sin \rho \left( \pi - \arctan \frac{n}{m} \right) \\
+ \beta \sin \rho \left( \arctan \frac{n}{m} \right) \right] \left( n^2 + m^2 \right)^{\rho/2}
\end{equation}

as $n \to \infty$ for two distinct rational values of $n/m$, then (14) remains true for all rational values of $n/m$, and as $R \to \infty$ we have

\[
\sum_{m=0}^{R} f(m, 0) \sim \alpha R^{1+\rho}, \quad \sum_{m=-R}^{0} f(m, 0) \sim \beta R^{1+\rho}.
\]

In (12) we define $g(x)$ to be constant in the interval $n < x < n+1$, for all integers $n$ and with saltus $f(n, 0)$ at $x = n$: then Theorems 2 and 3 follow directly from the corollary to Theorem 1 and Lemmas 4, 5, and 6.

Theorem 4. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, and for some finite $\bar{n} \geq 0$

\[
\lim_{m \to \infty} f(m, \bar{n}) = \alpha,
\]

then we have

\[
\lim_{m \to \infty} f(m, n) = \alpha_n
\]
for \( n \geq 0 \) where \( \alpha_n \) is a linear function of \( n \).

\( f(m, n) \) is non-negative and preharmonic in the half-plane \( n \geq \bar{n} \) and so by Theorem 1 has the representation

\[
f(m, n + \rho) = D\rho + \sum_{r=\infty}^{m} f(r, n)h(m - r, \rho)
\]

for \( \rho > 0 \), and some non-negative \( D \). From the definition of \( h(m, n) \) it is easily verified that

\[
\sum_{m=\infty}^{\infty} h(m, \rho) = 1.
\]

Also, from Lemma 3(d), for \( \rho > 0 \)

\[
h(m, \rho) < A\rho/(m^2 + \rho^2).
\]

From the hypothesis and Theorem 1, given \( \epsilon > 0 \) there exists an integer \( N > 0 \) for which

\[
| f(m, n) - \alpha | \leq \epsilon
\]

for \( m > N \) and for which

\[
\sum_{r=\infty}^{m-N} \frac{f(r, n)}{1 + r^2} \leq \epsilon.
\]

We may now apply (15)–(18) as follows:

\[
| f(m, n + \rho) - D\rho - \alpha |
\]

\[
\leq \sum_{r=\infty}^{m-N} | f(r, n) - \alpha | h(m - r, \rho)
\]

\[
\leq \sum_{r=\infty}^{m-N} f(r, n)h(m - r, \rho) + \alpha \sum_{r=\infty}^{m-N} h(m - r, \rho)
\]

\[
+ \sum_{r=N+1}^{\infty} | f(r, n) - \alpha | h(m - r, \rho)
\]

\[
\leq A\rho \sum_{r=\infty}^{m-N} \frac{f(r, n)}{1 + r^2} + \frac{A\rho}{(m - N)^2} \sum_{r=m-N}^{N} f(r, n)
\]

\[
+ A\rho \alpha \sum_{r=m-N}^{\infty} \frac{1}{r^2 + \rho^2} + \epsilon \sum_{r=N+1}^{\infty} h(m - r, \rho)
\]

\[
\leq A\rho \epsilon + \frac{A\rho}{(m - N)^2} \sum_{r=m-N}^{N} f(r, n) + \frac{A\rho \alpha}{m - N} + \epsilon.
\]
It is apparent that by a suitable choice of \( \varepsilon \) and correspondingly large \( m \), the right-hand side is arbitrarily small. This proves the theorem for \( n > \bar{n} \) and the complete result follows from the difference equation for preharmonic functions.

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References


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A THEOREM OF PHRAGMÉN-LINDELÖF TYPE

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1. Introduction. In the present paper the Phragmén-Lindelöf theorem for harmonic functions in the formulation of M. Heins [4] shall be extended to the solutions of the elliptic partial differential equation

\[
L_k[u] = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n} = 0 \quad (k < 1)
\]

(\( k \) denoting a real constant). Equation (1.1) appears in several problems. For an exposition of previous results in the theory of the solutions of (1.1) we refer to a recent paper of A. Weinstein [9].

A theorem of Phragmén-Lindelöf type for the solutions of a rather general class of elliptic partial differential equations has been proved by D. Gilbarg [3] and E. Hopf [5]. Because of the singular coefficient, (1.1) is not contained in this class.

We introduce the following notations, \( P(x_1, x_2, \ldots, x_n) \) denoting a point in the \( n \)-dimensional space:

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