A NOTE ON PREHARMONIC FUNCTIONS

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1. Let $L$ be the set of points whose coordinates are rational integers. Let $D$ be a domain, that is to say, an open connected set, and let $G$ be the set $D \cdot L$. A point $P(m, n)$ of $G$ is an interior point if the four points $(m \pm 1, n)$, $(m, n \pm 1)$ contiguous to $P$ belong to $G$. Otherwise $P$ is a boundary point.

A function $f(m, n)$ defined on $G$ is preharmonic if the value of $f$ at any interior point is the mean of the values of $f$ at the contiguous points, that is to say

$$4f(m, n) = f(m + 1, n) + f(m - 1, n) + f(m, n + 1) + f(m, n - 1).$$

For several decades the subject of preharmonic functions has been considered by many mathematicians, and the connection with harmonic functions has long been known. A recent paper by Heilbronn [1] states a number of theorems which are the analogues of classical theorems for harmonic functions.

In this note we consider functions which are preharmonic and non-negative in the half-plane $n \geq 0$ and prove a representation theorem analogous to that for positive harmonic functions [2], and a theorem which is the analogue of the Phragmén-Lindelöf type theorem for positive harmonic functions [3; 4].

2. We require the following lemmas:

**Lemma 1.** If $f(m, n)$ is preharmonic on a bounded domain $D$, then $f(m, n)$ is either constant or attains its maximum and minimum on $D$ on the boundary only.

**Lemma 2.** If $f(m, n)$ is preharmonic everywhere and satisfies the inequality

$$|f(m, n)| < A \{1 + (|m| + |n|)^k\}$$

for all $m, n$, where $k$ is a positive integer, then $f(m, n)$ is a polynomial of degree not exceeding $k$.

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1 In what follows, $A$ will always represent a positive nonzero number, independent of the variables in the context.
These lemmas are special cases of Theorems 1 and 6 of Heilbronn's paper.

**Lemma 3.** The function

\[ h(m, n) = \frac{1}{\pi} \int_{0}^{\pi} \cos mt \phi^n(t) dt, \]

where \( \phi(t) \) is the smaller root of the equation

\[ \phi(t) + \phi^{-1}(t) + 2 \cos t = 4, \]

is preharmonic everywhere with the following properties:

1. \( h(0, 0) = 1, \)
2. \( h(m, 0) = 0 \) for \( m \neq 0, \)
3. \( h(m, n) > 0 \) for \( n > 0, \)
4. \[ |h(m, n) - n/\pi(m^2 + n^2)| < A/n(m^2 + n^2) \text{ for all } m \text{ and positive } n. \]

(a) and (b) follow by inspection. To prove (c), let

\[ M(n) = \operatorname{glb}_{|m| < \infty} h(m, n) \]

for \( n \geq 0. \) It is easily seen that \( |\phi(t)| < 1 \) for \( 0 < t < \pi \) and so \( M(n) \to 0 \)

as \( n \to \infty \) and, from the difference equation for preharmonic functions, we have for \( n \geq 1 \)

\[ 2M(n) \geq M(n + 1) + M(n - 1) \]

and, since \( M(0) = 0, \) the result follows.

It may be verified that \( \phi(t) \) is a positive decreasing function of \( t \)
in \( (0, \pi) \) with derivatives of all orders there, that

(1) \( \phi'(\pi) = 0, \quad \lim_{t \to 0^+} \phi'(t) = -1, \)

(2) \( \phi(t) = 1 - t + t^2/2 - t^3/12 + O(t^4) \)
as \( t \to 0^+, \) and that there exists a real number \( \eta > 0 \) such that

(3) \( \phi(t) \leq e^{-\eta t} \quad \text{for } 0 \leq t \leq \pi. \)

Integrating by parts twice in the expression for \( h(m, n) \) we have from

(1) \( \text{and the fact that } \sin m\pi = 0 \)

\[ \pi h(m, n) = \frac{n}{m^2} - \frac{n}{m^2} \int_{0}^{\pi} \phi^{n-2}(t) \cos mt [(n - 1) \{ \phi'(t) \}^2 + \phi(t) \phi''(t)] dt, \]
or, adding \( \pi(n^2/m^2)h(m, n) \) to each side,
\[
\pi \cdot \frac{m^2 + n^2}{m^2} \cdot h(m, n) = \frac{n}{m^2} - \frac{n}{m^2} \int_0^\pi \phi^{n-2}(t)\psi(t)\cos mtdt,
\]

where

\[
\psi(t) = (n - 1)\left\{\phi'(t)\right\}^n + \phi(t)\phi''(t) - n\phi^2(t).
\]

From the enunciated properties of \(\phi(t)\) we may easily show that

\[
|\psi(t)| < A(|n| t^2 + 1)
\]

for \(0 < t \leq \pi\). Thus, by (3), we have for \(n \geq 1\)

\[
\left|\pi \cdot \frac{m^2 + n^2}{m^2} \cdot h(m, n) - \frac{n}{m^2}\right| < A \cdot \frac{n}{m^2} \int_0^\pi e^{-n^2(n^2t^2 + 1)}dt
\]

\[
< \frac{A}{nm^2},
\]

and this completes the proof of Lemma 3.

3. THEOREM 1. A necessary and sufficient condition for a function \(f(m, n)\) to be non-negative and preharmonic for \(n \geq 0\) is that the numbers \(f(m, 0) \{m = 0, \pm 1, \pm 2, \ldots\}\) should be non-negative and satisfy

\[
\sum_{m=-\infty}^{\infty} \frac{f(m, 0)}{1 + m^2} < \infty
\]

and that there should exist a non-negative number \(D\) for which

(4) \[f(m, n) = Dn + \sum_{r=-\infty}^{\infty} f(r, 0)h(m - r, n)\]

for \(n \geq 0\).

SUFFICIENCY. For any large positive \(N\) and \(n > 0\) we have, from Lemma 3(d),

\[
\sum_{r=-N}^{N} f(r, 0)h(m - r, n) < A \sum_{r=-N}^{N} \frac{f(r, 0)n}{(m - r)^2 + n^2}
\]

\[
< AC(m, n) \sum_{r=-N}^{N} \frac{f(r, 0)}{1 + r^2}
\]

where

\[
C(m, n) = \text{lub}_{|r| \leq n} \frac{n(1 + r^2)}{(m - r)^2 + n^2}
\]

and is finite for any fixed \(m, n\). Thus the function defined by (4) is
an absolutely convergent series of non-negative preharmonic functions and, hence, is itself non-negative and preharmonic for $n \geq 0$.

**Necessity.** Let $R$ be a positive integer and define

$$f_R(m, n) = f(m, n) - \sum_{r=-R}^{R} f(r, 0) h(m - r, n).$$

Evidently $f_R(m, n)$ is preharmonic in the half-plane $n \geq 0$ and also

$$f_R(m, n) \geq - \left\{ \max_{|r| \leq R} h(m - r, n) \right\} \sum_{r=-R}^{R} f(r, 0).$$

From Lemma 3(d) the right-hand side has arbitrarily small modulus for all points $(m, n)$ of the half-plane lying outside a sufficiently large circle with centre at $(0, 0)$. Since, by Lemma 1, a preharmonic function in a finite domain attains its minimum on the boundary and $f_R(m, n) = 0$ for $n = 0$, it follows that for $n \geq 0$, $f_R(m, n) \geq 0$. That is to say, for $n \geq 0$ we have

$$f(m, n) \geq \sum_{r=-R}^{R} f(r, 0) h(m - r, n),$$

and letting $R \to \infty$

$$f(m, n) \geq \sum_{r=-\infty}^{\infty} f(r, 0) h(m - r, n).$$

Next, by Lemma 3(d), there exists a large positive integer $N$ for which

$$h(m, N) > 1/(N^2 + m^2)$$

for all integers $m$. Thus we have, from (5),

$$f(0, N) \geq \sum_{r=-\infty}^{\infty} f(r, 0) h( - r, N)$$

\[ \geq \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{r^2 + N^2} \]

\[ \geq \frac{1}{N^2} \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2}. \]

This proves that if $f(m, n)$ is non-negative and preharmonic for $n \geq 0$,

$$\sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2} < \infty.$$
\[ f_\omega(m, n) = f(m, n) - \sum_{r=\infty}^{\infty} f(r, 0) h(m - r, n), \]

it remains to show that \( f_\omega(m, n) = Dn \) for some non-negative \( D \). Now since the series \( \sum_{r=\infty}^{\infty} f(r, 0) h(m - r, n) \) is convergent and each term is non-negative and preharmonic for \( n \geq 0 \), \( f_\omega(m, n) \) also is non-negative and preharmonic for \( n \geq 0 \), and, a fortiori, for any integral \( t > 0 \), \( f_\omega(m, n+t) \) is non-negative and preharmonic for \( n \geq 0 \). From what we have just proved above, we have

\[ \sum_{r=\infty}^{\infty} \frac{f_\omega(r, t)}{1 + r^2} < \infty \]

and, a fortiori, \( f_\omega(m, t) < \mathcal{K}_t(1 + m^2) \), where \( \mathcal{K}_t \) is finite for each integral \( t \). Let us assume for the moment that we have shown that

\[ f_\omega(m, n) < A n^2(1 + m^2) < A [1 + (|m| + |n|)^4] \]

for \( n > 0 \). We may continue \( f_\omega(m, n) \) uniquely throughout the entire plane by writing

\[ f_\omega(m, -n) = - f_\omega(m, n) \]

for \( n > 0 \), and have

(i) \( f_\omega(m, n) \) preharmonic everywhere,

(ii) \( f_\omega(m, n) < A [1 + (|m| + |n|)^4] \) everywhere,

(iii) \( f_\omega(m, 0) = 0 \) for all \( m \),

(iv) \( \text{sign} f_\omega(m, n) = \text{sign} n \) for \( n \neq 0 \).

Applying Lemma 2 to \( f_\omega(m, n) \) it follows from (8)(ii) that \( f_\omega(m, n) \) is a polynomial of degree not exceeding 4. From (8)(iii), \( n \) must be a factor of \( f_\omega(m, n) \); since \( f_\omega(m, n) \) by (7) contains only odd powers of \( n \) we must have

\[ f_\omega(m, n) = n \phi(m, n^2), \]

where \( \phi(m, n^2) \) is a polynomial of degree not exceeding 3. Further, from (8)(iv), \( \phi(m, n^2) \) is everywhere non-negative, and so of degree not exceeding 2. We have now shown that

\[ f_\omega(m, n) = n (\alpha m^2 + \beta n^2 + \gamma m + \delta) \]

where \( \alpha \) and \( \beta \) are non-negative. It may be verified, from the difference equation, that since \( f_\omega(m, n) \) is preharmonic, then \( \alpha + 3\beta = 0 \). Thus \( \alpha = \beta = 0 \) and this implies that \( \gamma = 0 \) and \( \delta \geq 0 \). This completes the proof that
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\( f_\omega(m, n) = Dn \)

for some non-negative \( D \).

It remains to prove (6). Consider the function

\[
g(m, n, m', 2n) = \frac{1}{2} \sum_{n=1}^{\infty} \sin \frac{\pi(n - m' + n)}{2n} \cdot \sin \frac{\pi \sinh \alpha \cdot n}{2 \sinh 2\alpha \cdot n}
\]

where \( \alpha \) is the positive root of the equation

\[
\cosh \alpha + \cos \frac{\pi}{2n} = 2.
\]

This function is preharmonic everywhere and may be shown to satisfy

\[
g(m, n, m', 2n) =
\begin{cases}
0 & \text{for } m = m' + n, \\
0 & \text{for } n = 0, \\
0 & \text{for } 1 \leq |m - m'| \leq n, n = 2n, \\
1 & \text{for } m = m', n = 2n.
\end{cases}
\]

Further,

\[
g(m, 1, m', 2n) = \frac{1}{2} \sum_{n=1}^{\infty} \sin \frac{\pi}{2n} \cdot \frac{\sinh \alpha}{\sinh 2\alpha \cdot n}
\]

From (9) we have

\[
\cosh \alpha = 2 - \cos \frac{\pi}{2n} < \cosh \frac{\pi}{2n},
\]

and so

\[
\alpha < \frac{\pi}{2n},
\]

and substituting this in the inequality for \( g(m, 1, m', 2n) \) we deduce that

\[
g(m, 1, m', 2n) > \frac{A}{n^2}.
\]

Let us suppose that (6) is not true. Then there exists an increasing sequence of integers \( \{n_i\} \), and a corresponding sequence of integers \( \{m_i\} \) such that as \( i \to \infty \)

\[\text{This method of writing preharmonic functions as a sum of products is due to Phillips and Wiener [5].}\]
We shall suppose first that the integers $n_r$ are even. Consider the function
\[ f_r(m, n) = f_m(m, n) - f_m(m, n)g(m, n, m, n_r). \]
From (10) and (11) it is apparent that $f_r(m, n) \geq 0$ on the boundary of the square $|m - m_1| \leq n_r$, $0 \leq n \leq n_r$, and also, by Lemma 1, inside the square. In particular, we have
\[ f_m(m, 1) \geq f_m(m, n_r)g(m, 1, m, n_r), \]
and so, by (11),
\[ f_m(m, n_r) < A n^2(1 + m^2), \]
which contradicts our assumption. Similarly, if the sequence is odd, we may show that
\[ f_m(m, n_r) < A n^2(1 + m^3). \]

**Corollary.** Suppose $f(m, n)$ to be preharmonic and non-negative in $n \geq 0$. Then, as $n\to\infty$ subject to the condition $am + bn = 0$ where $a$ and $b$ are integers,
\[ f(m, n) - H(m, n) = Dn + O\left\{n + n^2\right\}^{1/2}, \]
where $D$ is a non-negative number and
\[ H(m, n) = \frac{1}{\pi} \sum_{r=-\infty}^{\infty} f(r, 0) \cdot \frac{n}{(m - r)^2 + n^2}. \]
This result follows immediately from Theorem 1 and Lemma 3(d).

4. If $H(\rho\omega)$ is positive and harmonic in the half-plane $0<\theta<\pi$, then it may be written as [2]
\[ H(\rho\omega) = dr \sin \theta + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{r \sin \theta}{r^2 - 2rt \cos \theta + t^2} \cdot dg(t) \]
where $d$ is a non-negative number and $g(t)$ is a nondecreasing function such that
\[ \int_{-\infty}^{\infty} \frac{dg(t)}{1 + t^2} < \infty. \]

**Lemma 4.** If $H(\rho\omega)$ is defined by (12), $-1 < \rho < 1$, $0 < \phi < \pi$, $n$ is an integer and $\alpha$, $\delta$ are any positive numbers such that
as \( n \to \infty \), then

\[ H(n\delta e^{i\phi}) \sim a(n\delta)^p \]

as \( r \to \infty \).

There is no loss in generality in assuming \( d = 0 \), and so as \( n \to \infty \)

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dg(t)}{(n\delta)^2 - 2n\delta \cos \phi + t^2} \sim \alpha \cosec \phi \cdot (n\delta)^{-1}.
\]

This is easily shown to imply that for \( x > 1 \)

\[
\frac{g(x) - g(-x)}{x^2} + \int_x^{\infty} \frac{d\{g(t) - g(-t)\}}{x^2 + t^2} < Ax^{p-1}.
\]

Further, it will be sufficient to prove that for \(|r - \sigma| \leq 1\) and \( r \to \infty \)

\[ \mathcal{I}(re^{i\phi}) - H(\sigma e^{i\phi}) = o(r^p). \]

Now from (12), since \( d = 0 \),

\[
\left| H(re^{i\phi}) - H(\sigma e^{i\phi}) \right| = \left| \frac{(r - \sigma) \sin \phi}{\pi} \int_{-\infty}^{\infty} \frac{(t^2 - r\sigma)dg(t)}{t^2 - 2t\cos \phi + t^2} \right|
\]

\[
\leq A \left[ \frac{g(r) - g(-r)}{r^2} + \int_r^{\infty} \frac{d\{g(t) - g(-t)\}}{t^2 + 2} \right]
\]

\[
\leq Ar^{p-1} = o(r^p)
\]

as \( r \to \infty \).

The following two lemmas are contained in a paper by Allen and Kerr [4].

**Lemma 5.** If \( H(re^{i\phi}) \) is defined by (12), \(-1 < \rho < 1\), and

\[ H(re^{i\phi/2}) \sim (1 + \rho)ar^p \sec \rho x/2 \]

as \( r \to \infty \), then

\[ g(x). - g(-x) \sim 2ax^{1+p} \]

as \( x \to \infty \).

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\(^8\) Actually Allen and Kerr state their results for the case \( r \to 0^+ \), but the case \( r \to \infty \) is an elementary corollary.
Lemma 6. If $H(re^{\theta})$ is defined by (12), $-1 < \rho < 1$, and

$$H(re^{\theta}) \sim (1 + \rho) \csc \rho \pi \left[ \alpha \sin \rho (\pi - \theta) + \beta \sin \rho \theta \right] r^\sigma,$$

as $r \to \infty$ for two distinct values of $\theta$, then (13) remains true for all values of $\theta$ and as $x \to \infty$

$$g(x) - g(0) \sim \alpha x^{1+\sigma}, \quad g(0) - g(-x) \sim \beta x^{1+\sigma}.$$

Theorem 2. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, $-1 < \rho < 1$, and

$$f(0, n) \sim (1 + \rho) \alpha \sec \rho \pi/2 \cdot n^\sigma$$

as $n \to \infty$, then

$$\sum_{m=-R}^{R} f(m, 0) \sim 2\alpha R^{1+\sigma}$$

as $R \to \infty$.

Theorem 3. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, $-1 < \rho < 1$, and

$$f(m, n) \sim (1 + \rho) \csc \rho \pi \left[ \alpha \sin \rho \left( \pi - \arctan \frac{n}{m} \right) \right. 
+ \beta \sin \rho \left( \arctan \frac{n}{m} \right) \left( n^2 + m^2 \right)^{\sigma/2}$$

as $n \to \infty$ for two distinct rational values of $n/m$, then (14) remains true for all rational values of $n/m$, and as $R \to \infty$ we have

$$\sum_{m=-R}^{R} f(m, 0) \sim \alpha R^{1+\sigma}, \quad \sum_{m=-R}^{0} f(m, 0) \sim \beta R^{1+\sigma}.$$

In (12) we define $g(x)$ to be constant in the interval $n < x < n+1$, for all integers $n$ and with saltus $f(n, 0)$ at $x = n$; then Theorems 2 and 3 follow directly from the corollary to Theorem 1 and Lemmas 4, 5, and 6.

Theorem 4. If $f(m, n)$ is non-negative and preharmonic in the half-plane $n \geq 0$, and for some finite $n \geq 0$

$$\lim_{m \to \infty} f(m, n) = \alpha,$$

then we have

$$\lim_{n \to \infty} f(m, n) = \alpha_n.$$
for \( n \geq 0 \) where \( \alpha_n \) is a linear function of \( n \).

\( f(m, n) \) is non-negative and preharmonic in the half-plane \( n \geq \bar{n} \) and so by Theorem 1 has the representation

\[
f(m, \bar{n} + \rho) = D \rho + \sum_{r=0}^{\infty} f(r, \bar{n}) h(m - r, \rho)
\]

for \( \rho > 0 \), and some non-negative \( D \). From the definition of \( h(m, n) \) it is easily verified that

\[
\sum_{m=0}^{\infty} h(m, \rho) = 1.
\]

Also, from Lemma 3(d), for \( \rho > 0 \)

\[
h(m, \rho) < A \rho / (m^2 + \rho^2).
\]

From the hypothesis and Theorem 1, given \( \epsilon > 0 \) there exists an integer \( N > 0 \) for which

\[
|f(m, \bar{n}) - \alpha| \leq \epsilon
\]

for \( m > N \) and for which

\[
\sum_{r=-\infty}^{-N} \frac{f(r, \bar{n})}{1 + r^2} \leq \epsilon.
\]

We may now apply (15)—(18) as follows:

\[
|f(m, \bar{n} + \rho) - D \rho - \alpha|
\]

\[
\leq \sum_{r=0}^{N} |f(r, \bar{n}) - \alpha| h(m - r, \rho)
\]

\[
\leq \sum_{r=0}^{N} f(r, \bar{n}) h(m - r, \rho) + \alpha \sum_{r=0}^{N} h(m - r, \rho)
\]

\[
+ \sum_{r=N+1}^{\infty} |f(r, \bar{n}) - \alpha| h(m - r, \rho)
\]

\[
\leq A \rho \sum_{r=-\infty}^{-N} \frac{f(r, \bar{n})}{1 + r^2} + \frac{A \rho}{(m - N)^2} \sum_{r=-N}^{N} f(r, \bar{n})
\]

\[
+ A \rho \alpha \sum_{r=m-N}^{\infty} \frac{1}{r^2 + \rho^2} + \epsilon \sum_{r=N+1}^{\infty} h(m - r, \rho)
\]

\[
\leq A \rho \epsilon + \frac{A \rho}{(m - N)^2} \sum_{r=-N}^{N} f(r, \bar{n}) + \frac{A \rho \alpha}{m - N} + \epsilon.
\]
It is apparent that by a suitable choice of $\epsilon$ and correspondingly large $m$, the right-hand side is arbitrarily small. This proves the theorem for $n > \bar{n}$ and the complete result follows from the difference equation for preharmonic functions.

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REFERENCES


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A THEOREM OF PHRAGMÉN-LINDELÖF TYPE

ALFRED HUBER

1. Introduction. In the present paper the Phragmén-Lindelöf theorem for harmonic functions in the formulation of M. Heins [4] shall be extended to the solutions of the elliptic partial differential equation

\[ L_k[u] = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n} = 0 \quad (k < 1) \]

($k$ denoting a real constant). Equation (1.1) appears in several problems. For an exposition of previous results in the theory of the solutions of (1.1) we refer to a recent paper of A. Weinstein [9].

A theorem of Phragmén-Lindelöf type for the solutions of a rather general class of elliptic partial differential equations has been proved by D. Gilbarg [3] and E. Hopf [5]. Because of the singular coefficient, (1.1) is not contained in this class.

We introduce the following notations, $P(x_1, x_2, \ldots, x_n)$ denoting a point in the $n$-dimensional space:

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