CONCERNING CERTAIN TYPES OF WEBS
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In my dissertation\(^1\) I defined a \(W_n\) set as follows: If \(n > 1\), a \(W_n\) set is a compact continuum \(M\) for which there exists a family \(F\) of \(n\) elements such that (1) each element of \(F\) is an upper semicontinuous collection of mutually exclusive continua which fills up \(M\) and is an arc with respect to its elements, and (2) if \(G\) is a collection of continua each belonging to some, but no two to the same, collection of the family \(F\), then the continua of the collection \(G\) have a point in common and their common part is totally disconnected. In that paper it was stated\(^2\) without proof that in the plane there exists a \(W_2\) set \(M\) whose boundary is the point set \(B(M)\) and which has a complementary domain whose boundary, \(J\), contains six limit points of \(B(M) - J\) but that no \(W_3\) set has a complementary domain whose boundary contains seven such points. It is the purpose of this paper to prove this statement.

In what follows the space considered will be the plane and if \(M\) is a point set the notation \(B(M)\) will be used to denote its boundary.

THEOREM 1. There exists a \(W_2\) set \(M\) whose outer boundary, \(J\), contains six limit points of \(B(M) - J\).

Proof. Let \(J\) denote a circle with interior \(E\). Let \(Q_1, Q_2, \ldots, Q_6\) be six points of \(J\) in that order. Let \(\gamma_1, \gamma_2, \ldots, \gamma_6\) be six sequences of circles such that (1) for each \(i\) the sequence \(\gamma_i\) has a sequential limiting set which contains only the point \(Q_i\), and (2) if \(x\) and \(y\) are two circles each belonging to one of the \(\gamma_i's\) then \(x\) is exterior to \(y\) and \(x+y\) is a subset of \(E\). Let \(D\) denote a point set such that \(P\) is a point of it if and only if there is a circle of one of the \(\gamma_i's\) whose interior contains \(P\). Let \(M\) denote the point set \(J+E-D\). We shall prove that \(M\) is a \(W_2\) set.

Let \(C\) denote a circle which lies, together with its interior, \(I\), in \(M-B(M)\). There exist\(^3\) three collections of arcs, \(H_1, H_2,\) and \(H_3\), satisfying with respect to \(C+I\) all the requirements of the definition of a \(W_2\) set and such that (1) if \(h_i\) is an endelement of \(H_1, h_i, C\) is an

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\(^2\) Theorems 13 and 14.
\(^3\) This follows from the argument used in the proof of Theorem 2 of my dissertation.

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endpoint of $h_i$ and (2) if $h_i$ is a non-endelement of $H_i$, $h_i \cdot C$ consists of the endpoints of $h_i$. There is a reversible transformation of the number interval $[0, 1]$ into the collection $H_1$ such that if for each number $x$ in this interval $h_x$ denotes its image under this transformation and the number sequence $x_1, x_2, \cdots$ converges to the number $a$, then the limiting set of the sequence $h_x$, $h_{x_2}, \cdots$ is a subset of $h_a$. Let $P_0$ and $P_1$ denote $C \cdot h_0$ and $C \cdot h_1$ respectively. The circle $C$ is the sum of two arcs, $P_0XP_1$ and $P_0X'P_1$. If $0 < x < 1$, $h_x$ contains a point $P_x$ on $P_0XP_1$ and a point $P_x'$ on $P_0X'P_1$.

Let $\alpha_1$ denote a positive number and let $a$ and $b$ be two numbers between 0 and 1 such that $a$ is less than $b$. Let $Q_1Q_2$, $Q_2Q_3$, $Q_3Q_4$, $Q_4Q_5$, $Q_5Q_6$, and $Q_6Q_7$ be six nonoverlapping arcs whose sum is $J$. Let $u_a$ be an arc lying except for its endpoints in $M - J$ such that $u_a \cdot J = Q_1 + Q_2$, $u_a \cdot (C + I) = h_a$, $u_a - h_a$ is the sum of a countable number of straight line intervals with slope $\alpha_1$ or $-\alpha_1$ and the simple closed curve $u_a + Q_1Q_2$ encloses $h_0$ and every circle of $\gamma_1$ and $\gamma_2$ and neither encloses nor intersects any circle of the remaining $\gamma_i$'s. Let $u_b$ be an arc lying except for its endpoints in $M - (u_a + J)$ such that $u_b \cdot J = Q_4 + Q_5$, $u_b \cdot (C + I) = h_b$, $u_b - h_b$ is the sum of a countable number of straight line intervals with slope $\alpha_1$ or $-\alpha_1$, and $u_a + Q_1Q_2$ encloses $h_1$ and every circle of $\gamma_4$ and $\gamma_5$ and neither encloses nor intersects any element of $\gamma_1$ or $\gamma_6$.

Let $M_1$, $M_2$, and $M_3$ denote $u_a + Q_1Q_2$ plus its interior, $Q_2Q_3Q_4 + Q_4Q_5Q_6 + u_a + u_b$ plus its interior, and $u_a + Q_1Q_2$ plus its interior respectively. There is a continuous collection, $U_1$, of mutually exclusive arcs and simple closed curves filling up $M - M_1$ such that (1) $U_1$ is an arc with respect to its elements, (2) $h_0$ and $u_a + Q_1Q_2$ are the endelements of $U_1$, (3) if $x$ is an element of $\gamma_1$ or $\gamma_2$ and $u$ is an element of $U_1$ intersecting $x$ then $u \cdot x$ is totally disconnected and (4) if $u$ is a non-endelement of $U_1$, $u \cdot (C + I)$ is an element of $H_1$ and $u - u \cdot (C + I)$ is the sum of a countable number of straight line intervals with slope $\alpha_1$ or $-\alpha_1$.

There is a continuous collection, $U_2$, of mutually exclusive arcs and simple closed curves filling up $M_2 \cdot M$ such that (1) $U_2$ is an arc with respect to its elements, (2) $h_1$ and $u_b + Q_4Q_5$ are the endelements of $U_3$, (3) if $x$ is an element of $\gamma_4$ or $\gamma_5$ and $u$ is an element of $U_3$ intersecting $x$ then $u \cdot x$ is totally disconnected, and (4) if $u$ is a non-endelement of $U_3$, $u \cdot (C + I)$ is an element of $H_1$ and $u - u \cdot (C + I)$ is the sum of a

\footnote{It follows from Theorem 8 of my dissertation that if $U$ is the collection of all elements of $U_1$ intersecting $x$, then the endelements of $U$ are simple closed curves, the non-endelements of $U$ are arcs, and no non-endelement intersects any element of $\gamma_1$ or $\gamma_2$ other than $x$.}
countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\).

There is an upper semicontinuous collection, \( U_2 \), of mutually exclusive continuous curves filling up \( M \) such that (1) \( U_2 \) is an arc with respect to its elements, (2) \( u_b \) and \( u_a \) are the endelements of \( U_2 \), (3) if \( u \) is an element of \( U_2 \) it contains only one point of the arc \( Q_2Q_3Q_4 \) and only one point of the arc \( Q_5Q_6Q_7 \), the point set \( u \cdot (C+I) \) is an element of \( H_1 \) and \( u-u \cdot (C+I) \) is either the sum of a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\) or the sum of some elements of \( \gamma_3 \) or \( \gamma_6 \) and a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), and (4) if \( u \) is an element of \( U_2 \) and it intersects an element of \( \gamma_3 \) or \( \gamma_6 \) it contains that element.

Let \( G_x \) denote the sum of the collections \( U_1, U_2, \) and \( U_3 \). The collection \( G_1 \) is an upper semicontinuous collection of mutually exclusive continuous curves filling up \( M \) such that \( G_1 \) is an arc with respect to its elements and each element of \( G_1 \) is either (1) an element of \( H_1 \), (2) the sum of an element of \( H_1 \) and a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), or (3) the sum of an element of \( H_1 \), a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), and either \( Q_1Q_2, Q_3Q_4, \) or some elements of \( \gamma_3 \) or \( \gamma_6 \).

If, in the above, we replace \( \alpha \) by a positive number \( \alpha_2 \) different from \( \alpha \) and if we replace \( \gamma_3 \) by \( \gamma_2 \), we can obtain an upper semicontinuous collection, \( G_2 \), of mutually exclusive continuous curves filling up \( M \) such that \( G_2 \) is an arc with respect to its elements and each element of \( G_2 \) is either (1) an element of \( H_2 \), (2) the sum of an element of \( H_2 \) and a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), or (3) the sum of an element of \( H_2 \), a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), and either \( Q_3Q_4, Q_5Q_6, \) or some elements of \( \gamma_2 \) or \( \gamma_1 \).

Replacing \( \alpha \) by a positive number \( \alpha_3 \) different from \( \alpha \) and \( \alpha_2 \) and replacing \( \gamma_3 \) by \( \gamma_2 \) we can obtain an upper semicontinuous collection, \( G_3 \), of mutually exclusive continuous curves filling up \( M \) such that \( G_3 \) is an arc with respect to its elements and each element of \( G_3 \) is either (1) an element of \( H_3 \), (2) the sum of an element of \( H_3 \) and a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), or (3) the sum of an element of \( H_3 \), a countable number of straight line intervals with slope \( \alpha \) or \(-\alpha\), and either \( Q_3Q_4, Q_5Q_6, \) or some elements of \( \gamma_2 \) or \( \gamma_1 \).

The collections \( G_1, G_2, \) and \( G_3 \) satisfy with respect to \( M \) all the requirements of the definition of a \( W_3 \) set.

**Theorem 2.** If \( M \) is a \( W_3 \) set and \( J \) is the boundary of a complementary domain of \( M \), then \( J \) does not contain seven limit points of \( B(M) - J \).
Proof. The continuum $J$ is a simple closed curve. Suppose $J$ does contain seven limit points of $B(M) - J$. Let $\alpha$ denote a collection of seven such points of $J$. The simple closed curve $J$ is the sum of seven nonoverlapping arcs the sum of whose endpoints is the sum of the elements of $\alpha$. Let $\beta$ denote the collection of these arcs.

Let $G_1, G_2,$ and $G_3$ be collections satisfying with respect to $M$ all the requirements of the definition of $W_3$ set. For each $i$ less than 4 let $H_i$ denote the collection of all elements of $G_i$ which intersect $J$. We shall prove first that each of these collections consists of more than one element.

Suppose $H_3$ consists of only one element, $h$. Since $J$ is a subset of $h$ and $M$ is a $W_3$ set with respect to $G_1, G_2,$ and $G_3$, each of the collections $H_1$ and $H_2$ contains more than one element. Consequently, for each $i$ less than 3, $H_i$ contains two elements, $h_i$ and $h_i'$, neither of which separates $H_i^*$.\(^6\) Let $U_i$ denote $H_i^* - (h_i + h_i')$. Since $h_i \cdot J$ and $h_i' \cdot J$ are connected and $h_i \cdot h$ and $h_i' \cdot h$ are totally disconnected, $h_i \cdot J$ and $h_i' \cdot J$ are degenerate. Consequently $U_1$ and $U_2$ each contains five points of $\alpha$ and one of these points is in $U_1 \cdot U_2$. Let $P$ denote one such point.

Since $U_1$ and $U_2$ are open subsets of $M$ and $P$ is a limit point of $B(M) - J$, there is a complementary domain of $M$ whose boundary, $J'$, intersects $U_1 \cdot U_2$. The continuum $J'$ is therefore a subset of an element of $H_1$ and an element of $H_2$, which is contrary to the assumption that $M$ is a $W_3$ set with respect to $G_1, G_2,$ and $G_3$. Hence $H_3$ contains more than one element. Let $h_3$ and $h_3'$ be the elements of $H_3$ which do not separate $H_3^*$ and let $U_3$ denote $H_3^* - (h_3 + h_3')$.

For each $i$ less than 4 let $k_i$ denote the number of points of $\alpha$ lying in $U_i$. There are $7-k_i$ points of $\alpha$ in $h_i + h_i'$ and if $k_i$ is not greater than 4 then each of at least $7-k_i-2$ arcs of $\beta$ is a subset of $h_i$ or $h_i'$. Let $l_i$ denote $5-k_i$. If $k_i$ exceeds 4 let $l_i$ be 0. In any case, $5-k_i \leq l_i$. Since no point of $\alpha$ is common to two of the point sets $U_1, U_2,$ and $U_3$, $k_1 + k_2 + k_3 \leq 7$. Since no arc of $\beta$ is a subset of two elements of the sum of the collections $H_1, H_2,$ and $H_3$, $l_1 + l_2 + l_3 \leq 7$. However, $l_1 + l_2 + l_3 + (5-k_1) + (5-k_2) + (5-k_3) \geq 8$. Thus the assumption that $J$ contains seven limit points of $B(M) - J$ leads to a contradiction.

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\(^7\) See Theorem 8 of my dissertation.

\(^8\) See the corollary to Theorem 8 of my dissertation.

\(^9\) This follows from an argument similar to that used to prove that $H_3$ does not contain only one element.
In a great many cases the methods used in the proofs of the above theorems can be used to determine whether a given continuum is a $W_n$ set. In particular, they can be used to prove that no $W_1$ set, $M$, has a complementary domain whose boundary, $J$, contains three limit points of $B(M) - J$, no $W_4$ set has a complementary domain whose boundary contains five such points, and that there exists a $W_6$ set whose outer boundary contains three such points.

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ANOTHER REMARK ON "SOME PROBLEMS IN
CONFORMAL MAPPING"

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It was remarked in [2] and proved in [3] that for every triply-connected domain $D$ there are certain triply-connected subdomains $D'$ having the same topological situation and admitting no conformal mapping into $D$ preserving this topological situation other than the identity. This result implies at once several others. Indeed let $D$ have contours $K_1, K_2, K_3$ and let the corresponding contours of $D'$ be $K'_1, K'_2, K'_3$. It is assumed no contour of $D$ reduces to a point. If $D'$ is obtained from $D$ by producing slits from $K_2, K_3$ out onto the same connected piece of the line of symmetry of $D$, it is clear that there is no conformal mapping of $D'$ into $D$ which can make $K'_1$ go into $K_2$ or $K'_1$ go into $K_3$ (in the natural sense of boundary correspondence). Thus for a domain $D$ and subdomain $D'$ there may exist no conformal mapping of the above type which carries either (a) a given boundary contour of $D'$ into the corresponding boundary contour of $D$ or (b) some two boundary contours of $D'$ into the corresponding two boundary contours of $D$.

The question naturally arises whether given a triply-connected domain $D$ and a triply-connected subdomain $D'$ having the same topological situation there exists a conformal mapping of $D'$ into $D$ preserving the topological situation and carrying some one contour of $D'$ into the corresponding contour of $D$. This question was raised to me by Professor A. Beurling some four or five years ago. The simple example above is not sufficient to provide an answer since in it

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