Theorem 10. For the derivatives of functions \( f_p(z) \) of class \( S \):
\[
1 - |z| \leq |f'_p(z)| \leq 1 + |z|.
\]
These inequalities are sharp. They are attained by \( f_p(z) = z - z^2/2 \) at \( z = \pm r, \ r \) real.

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A REMARK ON REVERSIBLE MATRICES

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The matrix \( A = (a_{nk}) \) is called reversible if for each \( y = \{y_n\} \in (c) \) the equations \( y_n = \sum_{k=0}^\infty a_{nk}x_k \ (n = 0, 1, \ldots) \) have exactly one solution \( x = \{x_k\} \). In this case there exist \([1, 4]\) constants \( c_k, b_{kn} \) with \( \sum_n |b_{kn}| < \infty \) for each \( k \), such that

\[
x_k = c_k \lim_{n \to \infty} y_n + \sum_{n=0}^\infty b_{kn}y_n \quad (k = 0, 1, \ldots).
\]

It is further stated in \([1, p. 50]\) that the \( c_k \) are bounded. This is questioned in \([4]\), where it is pointed out that if the \( c_k \) were generally bounded they would have to be almost all zero, but this remark does not dispose of the matter, for it might conceivably be a true theorem that for each reversible matrix the \( c_k \) are almost all zero. (For row-finite matrices, all \( c_k \) vanish; \([3]\).) The example given in \([4, p. 47]\) seems inconclusive. The purpose of this note is to show by a very simple example that in fact the \( c_k \) need not be bounded.

Consider the transformation
\[
y_{2m} = \sum_{p=0}^m x_{2p}, \quad y_{2m+1} = 2^{-m}x_{2m+1} + \sum_{p=0}^\infty x_{2p},
\]
where \( m = 0, 1, \ldots \). For each \( y \in (c) \) we have
\[
x_{2m+1} = 2^m(y_{2m+1} - \lim_n y_n);
\]
thus \( c_{2m+1} = -2^m \) is not bounded.

This has a bearing on a paper \([2]\) in which the following theorem is stated:

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Theorem. A reversible conservative matrix A sums a divergent sequence if and only if \( \| A^{-1} \| = \infty \).

Here \( \| A^{-1} \| \) is defined as \( \sup_k (|c_k| + \sum_n |b_{kn}|) \) with \( c_k, b_{kn} \) as above, and the theorem is proved under the assumption that the \( c_k \) are bounded. There is, however, no difficulty in allowing for the possibility of the \( c_k \) being unbounded. For then in the sufficiency part of the proof [2, p. 915], we note that if \( \| B \| = \sup_k \sum_n |b_{kn}| < \infty \), we may take \( y = \{1, 1, 1, \ldots\} \), and obtain at once from (1) a divergent sequence \( x \) which is \( A \)-summable, while if \( \| B \| = \infty \), we can find by classical methods a sequence \( y = \{y_n\} \) with \( \lim_n y_n = 0 \) and \( By \) unbounded. For such a sequence the \( c_k \) are immaterial and we have simply \( x = By \). This completes the proof.

References


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