If $u(x, t)$ is the temperature of an infinite insulated rod, then

$$H(a, b; t) = \int_a^b u(x, t) \, dx$$

is the amount of heat at time $t$ in the interval $[a, b]$. We prove an existence and uniqueness theorem for the case in which the heat, rather than the temperature, is prescribed initially. If the prescribed heat is not absolutely continuous, its derivative $\partial H / \partial b$, the initial temperature, may not determine $H$, so that we are dealing with a more general problem than the usual one.

The following lemma, which we did not find in the literature, will be useful.

**Lemma 1.** Let $S(x)$ be measurable, and $|f(x)| \leq M e^{\alpha x}$. Define

$$W_r(x, t) = (4\pi t)^{-1/2} \frac{\partial^r}{\partial x^r} [e^{-x^2/4t}].$$

Then, for every integer $r \geq 0$, the integral

$$u_r(x, t) = \int_{-\infty}^{\infty} W_r(x - \xi, t) f(\xi) \, d\xi$$

is absolutely convergent and satisfies

(a) $\frac{\partial u_r}{\partial x} = u_{r+1}$, $\frac{\partial u_r}{\partial t} = u_{r+2}$ on $0 < t < 1/4c$,

and, for all $t_0$ satisfying $0 < t_0 < 1/4c$,

(b) $|u_r(x, t)| \leq K t^{-r/2} e^{N x^2}$, on $0 < t \leq t_0$,

where $K = K(r, t_0)$, and $N = N(t_0)$.

**Proof.** It is well known that $W_r(x, t)$ is analytic, satisfies the heat equation, and has the specific form

$$W_r(x, t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{(2t)^{(r+1)/2}} H_r \left( \frac{x}{(2t)^{1/2}} \right) e^{-x^2/4t},$$

Received by the editors June 26, 1953.

Work done at Harvard University under Contract N5ori-07634, with the Office of Naval Research.
where $H_r(z)$ is the Hermite polynomial of degree $r$ in $z = x/(2t)^{1/2}$. The absolute convergence of (3) on $0 < t < 1/4c$ is obvious from (4) and $|f(x)| \leq Me^{ct^2}$.

To prove the first equation of (a), note

$$h^{-1}[u_r(x + h, t) - u_r(x, t)] = \int_{-\infty}^{\infty} W_{r+1}(x + \theta h - \xi, t) f(\xi) d\xi,$$

$$0 \leq \theta \leq 1,$$

by the Law of the Mean. But, as $|\xi| \to \infty$,

$$|W_{r+1}(x + \theta h - \xi, t)| = O(1 - \xi| + 1)^{r+1}e^{-(1-\xi^2)/4t),$$

and is bounded in finite intervals. Hence, since $t < 1/4c$, the integrand in (5) tends dominatedly [3, p. 168] to $W_{r+1}(x - \xi, t)f(\xi)$ as $h \to 0$, which shows that

$$\lim_{h \to 0} h^{-1}[u_r(x + h, t) - u_r(x, t)] = \int_{-\infty}^{\infty} W_{r+1}(x - \xi, t)f(\xi) d\xi,$$

proving the first equation of (a). A similar proof applies to the second equation.

To prove (b), note that the right side of (3) is by (4) bounded by a sum of terms of the form

$$Mt^{-s} \int_{-\infty}^{\infty} |x - \xi|e^{-(x-\xi)^2/4t} + c\xi^2 d\xi$$

Completing the square, the exponent becomes

$$[(1 - 4ct)/4t][\xi - x/(1 - 4ct)]^2 + cx^2/(1 - 4ct).$$

Writing $\eta = t^{-1/2}[\xi - x/(1 - 4ct)]$, and noting that $c/(1 - 4ct) < c/(1 - 4ct_0) = N'(t_0)$, we bound the right side of (3) by a sum of terms

$$M'e^{N'\xi^2} \int_{-\infty}^{\infty} |\eta t + N'xt| e^{-(1-4ct)s^2} d\eta.$$

Since $1 - 4ct \geq 1 - 4ct_0 = L > 0$, another bound is

$$M'e^{N'\xi^2} \int_{-\infty}^{\infty} |\eta + N'xt^{1/2}| e^{-L\eta} d\eta.$$

For any $N > N'(t_0)$ and suitable $K = K(t_0, r)$, this evidently implies (b).

**Corollary.** In Lemma 1, $u_r \in C^\infty$ and $\partial^{m+n}u_r/\partial x^m \partial t^n = u_{r+m+n}$. 

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We shall also use the following essentially known facts.

**Theorem A.** Let \( f(x) \) be measurable, and let \( |f(x)| \leq Me^{cx}, \ c > 0 \). Then

\[
(6) \quad u(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t)f(\xi)d\xi,
\]

is absolutely convergent for \( 0 < t < 1/4c \), and

\[
(\alpha) \quad u(x, t) \in C^\infty \text{ and } u_{xx} = u_t, \quad 0 < t < 1/4c,
\]

\[
(\beta) \quad |u(x, t)| \leq Ke^{Nz^2}, \quad 0 < t \leq t_0 < 1/4c, \quad K = K(t_0), \quad N = N(t_0).
\]

If \( f(x) \) has bounded variation on every finite interval, and \( 2f(x) = f(x^+) + f(x^-) \), then

\[
(\gamma) \quad \lim_{t \to 0^+} u(x, t) = f(x).
\]

If \( f(x) \) is continuous, then

\[
(\gamma') \quad \lim_{t \to 0^+, x \to x_0} u(x, t) = f(x_0),
\]

so that if \( u(x, 0) \) is defined as \( f(x) \), then \( u(x, t) \) is continuous for \( 0 \leq t < 1/4c \).

**Proof.** For \((\alpha), (\gamma), \) and \((\gamma')\) see [1, p. 298]; \((\beta)\) is part of Lemma 1.

**Theorem B.** Conversely, if \( u(x, t) \) is defined for \( 0 < t \leq t_0 \) and satisfies

\[
(\alpha) \quad u(x, t) \in C^2 \text{ and } u_{xx} = u_t, \quad 0 < t \leq t_0,
\]

\[
(\beta) \quad |u(x, t)| \leq Ke^{Nz^2}, \quad 0 < t \leq t_0, \quad \text{for some finite constants}
\]

\[
K = K(t_0), \quad N = N(t_0),
\]

\[
(\gamma) \quad \lim_{t \to 0^+} u(x, t) = 0, \quad \text{for all } x,
\]

then

\[
u(x, t) \equiv 0, \ 0 < t \leq t_0.
\]

The usual theorem, assuming simultaneous continuity in \( x \) and \( t, t \geq 0 \), is in [2, p. 88]. To prove our slightly sharper result, note that by the usual theorem, if \( 0 < \tau < t < 1/4N \), then

\[
u(x, t) = \lim_{t \to 0^+} \int_{-\infty}^{\infty} u(\xi, \tau)W_0(x - \xi, t - \tau)d\xi.
\]

If \( \tau < t/2 \), then \( W_0(x - \xi, t - \tau) \leq (2\pi t)^{-1/2}e^{-\frac{(x-\xi)^2}{2t}} \). Hence \((\beta)\) gives dominated convergence [3, p. 168], and we can pass to the limit
under the integral sign. We conclude that \( u(x, t) = 0 \) for all \( t \) such that \( t < 1/4N \). Suppose \( t_1 \) is the l.u.b. of the values of \( t \) for which \( \tau < t \rightarrow u(x, \tau) = 0 \), and suppose \( t_1 < t_0 \). Then, by continuity, \( u(x, t_1) = 0 \), and we may repeat the preceding argument to prove that \( u(x, t) \equiv 0 \), \( t \leq t_1 + \eta \), for some positive \( \eta \). This contradiction shows that \( t_1 = t_0 \), completing the proof of Theorem 2.

We shall now prove our existence theorem.

**Theorem 1.** Let \( \beta(x) \) have bounded variation on every finite interval, let \( 2H(x) = H(x^+) + H(x^-) \), \( H(0) = 0 \), and let \( |\beta(x)| \leq Me^{\alpha x} \). Then the improper integral

\[
(7) \quad u(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t)dH(\xi) = \lim_{A \to \infty} \int_{-A}^{A} W_0(x - \xi, t)dH(\xi)
\]

exists for \( 0 < t < 1/4c \), and satisfies

(a) \( u(x, t) \in C^\infty \) and \( u_{xx} = u_t \) on \( 0 < t < 1/4c \),

(b) \( |u(x, t)| \leq Kt^{-1/2}e^{Nz^2} \) on \( 0 < t \leq t_0 < 1/4c \),

(c) \( \lim_{t \to 0^+} \int_0^z u(\xi, t)d\xi = H(x) \) for all \( x \),

(d) \( \int_0^z u(\xi, t)d\xi \leq Le^{Nz^2} \) on \( 0 < t \leq t_0 < 1/4c \),

where \( L = L(t_0) \) and \( N = N(t_0) \).

**Proof.** If \( 0 < t < 1/4c \), then by direct calculation,

\[
u(x, t) = \lim_{A \to \infty} \int_{-A}^{A} W_0(x - \xi, t)dH(\xi)
= \lim_{B \to \infty} \left\{ [W_0(x - \xi, t)H(\xi)]_{-A}^{A} + \int_{-A}^{A} W_1(x - \xi, t)H(\xi)d\xi \right\}.
\]

By (4) and \( |H(x)| \leq Me^{\alpha x} \), the term in square brackets tends to zero. Hence

\[
u(x, t) = \int_{-\infty}^{\infty} W_1(x - \xi, t)H(\xi)d\xi.
\]

Lemma 1 assures us that (8) converges absolutely for \( 0 < t < 1/4c \), and has the properties (a), (b). Hence we can interpret \( u(x, t) \) as coming from an initial dipole distribution of density \( H(x) \).

To prove (c), integrate (8). Defining
(9) \[ H(x, t) = \int_0^x u(\xi, t) d\xi, \]

we get

(9') \[ H(x, t) = \int_{-\infty}^{\infty} \left[ W_0(x - \xi, t) - W_0(-\xi, t) \right] H(\xi) d\xi, \]

where the Fubini theorem is used to interchange the order of integration. Letting \( t \) tend to zero, and using (7) of Theorem A, we get (c). To prove (d), we apply (β) of Theorem A to both terms of (9') and add.

To prove uniqueness, we shall want the following result.

**Theorem C.** If \( u(x, t) \in C^2, u_{xx} = u_t, \) and \( |H(x, t)| \leq Ke^{Nz^2}, \) where \( H(x, t) \) is defined by (9) for all \( 0 \leq t \leq T, \) then

(10) \[ u(x, t) = \int_{-\infty}^{\infty} W_1(x - \xi, t) H(\xi, 0) d\xi, \quad \text{for } t < 1/4N. \]

**Proof.** We first note that

\[ H_t = \int_0^x u_t(\xi, t) d\xi = \int_0^x u_{xx}(\xi, t) d\xi = u_x(x, t) - u_x(0, t), \]

\[ H_{xx} = u_x(x, t). \]

Hence, if we define \( \overline{H}(x, t) = H(x, t) + \int_0^t u_x(0, \tau) d\tau, \) then \( \overline{H}(x, t) \) satisfies the heat equation, \( \overline{H}_{xx} = \overline{H}_t. \) Also \( \overline{H}(x, t) \in C^2 \) and \( |\overline{H}(x, t)| \leq K'te^{Nz^2}. \) Hence Theorem B applies, and

\[ \overline{H}(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) \overline{H}(\xi, 0) d\xi = \int_{-\infty}^{\infty} W_0(x - \xi, t) H(\xi, 0) d\xi. \]

Using part (a) of Lemma 1, we differentiate with respect to \( x \) and obtain (10), q.e.d.

**Theorem 2.** Let \( u(x, t) \in C^2, u_t = u_{xx}, \) \( |H(x, t)| \leq Ke^{Nz^2}, \) where \( H(x, t) \) is defined by (9), for all \( 0 < t \leq T, \) and \( H(x, t) \to 0 \) as \( t \to 0^+ \) for almost all \( x. \) Then \( u(x, t) \equiv 0 \) on \( 0 < t \leq T. \)

**Proof.** Let \( 0 < \tau < t < \min \{1/4N, T\}, \) and \( u_1(x, t - \tau) = u(x, t), \)

\( H_1(x, t - \tau) = H(x, t). \) Then for \( t \geq \tau, \) \( u_1(x, t - \tau), \) \( H_1(x, t - \tau) \) satisfy the conditions of Theorem C. Hence

\[ u_1(x, t - \tau) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau) H_1(\xi, 0) d\xi, \]
or

\[ u(x, t) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau) H(\xi, \tau) d\xi. \]

By hypothesis \( H(\xi, \tau) \to 0 \) almost everywhere dominatedly as \( \tau \to 0 \). Hence, using the Lebesgue dominated convergence theorem again, we have \( u(x, t) = 0 \), \( 0 < t < \min [1/4A, T] \). If \( 1/4A < T \), we repeat the argument, considering \( \tilde{u}(x, t) = h(x, t + 1/4N) \) instead of \( u(x, t) \): \( \tilde{u}(x, 0) = 0 \) by the continuity of \( u(x, t) \) in \( 0 < t < T \). This shows \( u(x, t) = 0 \), \( 0 < t < \min [1/4N, T] \). The proof is complete after at most \([4NT]\) steps.

**Corollary.** Under the hypotheses of Theorem 1, \( u(x, t) \) as defined by (6) is the only function satisfying conditions (a), (c), (d).

**Bibliography**


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\(^2\) J. L. B. Cooper (J. London Math. Soc. vol. 25 (1950) pp. 173–180) has proved a result related to Theorem 2. However, he requires that \( \int_a^b g(x) \delta x \to 0 \) for all bounded integrable \( g(x) \); we require only \( \int_a^b u(x, t) dx \to 0 \), for all \( a, b \). Cf. also J. Kampé de Fériet, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1527–1929.