A NOTE ON SEMI-GROUPS OF UNBOUNDED SELF-ADJOINT OPERATORS

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In this note we extend a theorem about integral representations of semi-groups of bounded self-adjoint operators to a theorem about semi-groups of unbounded self-adjoint operators. The theorem for semi-groups of bounded self-adjoint operators has been proved in different ways by B. v. Sz. Nagy [**4**; **5**], E. Hille [**3**], and also follows from a more general theorem of S. Bochner [**1**]. Semi-groups of unbounded self-adjoint operators arose in quite a natural way in an investigation of the author [**2**] on positive definite functions.

In the following theorem D_A shall represent the domain of an operator A which is defined in a Hilbert space.

THEOREM. *Let $\{T_x\}$, for $x>0$, be a semi-group of self-adjoint operators acting in a Hilbert space, i.e. $T_{x_1}T_{x_2}=T_{x_1+x_2}$. Further, suppose that for all $f\in\bigcap_{x>0}D_{T_x}$, (T_xf,f) is either a bounded or measurable function of x in some interval. Then there exists a unique resolution of the identity $\{E_t\}$ such that $E_t=0$ for $t<0$ and*

$$T_x = \int_0^\infty t^x dE_t.$$

PROOF. It has been shown by Sz. Nagy [**5**, p. 73] that the conditions on (T_xf,f) given in the theorem imply that this is a continuous function of x for all $x>0$.

Let us also use one more idea due to Sz. Nagy [**5**, p. 73]. Let $\{E_t\}$ be the canonical resolution of the identity of T_1. By the semi-group property we know that $T_1\geqq 0$. Since every positive self-adjoint operator has a unique positive self-adjoint square root, we have

$$T_{1/2} = \int_0^\infty t^{1/2} dEt.$$

Proceeding in this way and using the semi-group property we must get for all positive integers m and n that

$$T_{m/2^n} = \int_0^\infty t^{m/2^n} dE_t.$$

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Let us now consider the operators

$$H_x = \int_0^\infty t^x dE_t.$$

By the discussion above $H_{m/2^n} = T_{m/2^n}$. Now, there exists a countable set of mutually orthogonal manifolds $\{M_k\}_1^\infty$, whose direct sum is the whole space and such that for all $x>0$, $H_x = \prod_{k=1}^\infty \times H_x^{(k)}$, where $H_x^{(k)}$ is a bounded self-adjoint operator on M_k and is the restriction of H_x to M_k (Sz. Nagy [5, p. 49]). That is to say, $f \in D_{H_x}$ if and only if $\sum_{k=1}^\infty \|H_x^{(k)} f_k\|^2 < \infty$, where $f_k \in M_k$ and $f = \sum_{k=1}^\infty f_k$. Then $H_x = \sum_{k=1}^\infty H_x^{(k)} f_k$.

Given any $x>0$ there exists a rational number $m/2^n \geqq x$. From the semi-group property we know then that $D_{T_{m/2^n}} \subseteq D_{T_x}$. Therefore for every $f_k \in M_k$, $f_k \in D_{T_{m/2^n}}$ and therefore belongs to D_{T_x}. Consequently, since $T_{m/2^n} = H_{m/2^n}$ and by the continuity of $(T_x f_k, f_k)$ and $(H_x f_k, f_k)$ as functions of x, we must have $T_x f_k = H_x^{(k)} f_k = H_x f_k$. This implies that $T_x = H_x$ (Sz. Nagy [5, p. 35]).

The theorem for two parameter semi-groups offers no difficulty.

References

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