A NOTE ON SEMI-GROUPS OF UNBOUNDED SELF-ADJOINT OPERATORS

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In this note we extend a theorem about integral representations of semi-groups of bounded self-adjoint operators to a theorem about semi-groups of unbounded self-adjoint operators. The theorem for semi-groups of bounded self-adjoint operators has been proved in different ways by B. v. Sz. Nagy [4; 5], E. Hille [3], and also follows from a more general theorem of S. Bochner [1]. Semi-groups of unbounded self-adjoint operators arose in quite a natural way in an investigation of the author [2] on positive definite functions.

In the following theorem $D_A$ shall represent the domain of an operator $A$ which is defined in a Hilbert space.

Theorem. Let $\{T_x\}$, for $x>0$, be a semi-group of self-adjoint operators acting in a Hilbert space, i.e. $T_{x_1}T_{x_2} = T_{x_1+x_2}$. Further, suppose that for all $f \in \mathbb{D}_T$, $(T_x f, f)$ is either a bounded or measurable function of $x$ in some interval. Then there exists a unique resolution of the identity $\{E_t\}$ such that $E_t = 0$ for $t<0$ and

$$T_x = \int_0^\infty t^2 dE_t.$$ 

Proof. It has been shown by Sz. Nagy [5, p. 73] that the conditions on $(T_x f, f)$ given in the theorem imply that this is a continuous function of $x$ for all $x>0$.

Let us also use one more idea due to Sz. Nagy [5, p. 73]. Let $\{E_t\}$ be the canonical resolution of the identity of $T_1$. By the semi-group property we know that $T_1 \geq 0$. Since every positive self-adjoint operator has a unique positive self-adjoint square root, we have

$$T_{1/2} = \int_0^\infty t^{1/2} dE_t.$$ 

Proceeding in this way and using the semi-group property we must get for all positive integers $m$ and $n$ that

$$T_{m/n} = \int_0^\infty t^{m/n} dE_t.$$ 

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Let us now consider the operators

\[ H_x = \int_0^\infty t^x \, dE_t. \]

By the discussion above \( H_{m/2^n} = T_{m/2^n} \). Now, there exists a countable set of mutually orthogonal manifolds \( \{M_k\}_{k=1}^\infty \), whose direct sum is the whole space and such that for all \( x > 0 \), \( H_x = \prod_{k=1}^\infty H_x^{(k)} \), where \( H_x^{(k)} \) is a bounded self-adjoint operator on \( M_k \) and is the restriction of \( H_x \) to \( M_k \) (Sz. Nagy [5, p. 49]). That is to say, \( f \in D_{H_x} \) if and only if \( \sum_{k=1}^\infty \|H_x^{(k)}f_k\|^2 < \infty \), where \( f_k \in M_k \) and \( f = \sum_{k=1}^\infty f_k \). Then \( H_x = \sum_{k=1}^\infty H_x^{(k)} f_k \).

Given any \( x > 0 \) there exists a rational number \( m/2^n \geq x \). From the semi-group property we know then that \( D_{T_{m/2^n}} \subseteq D_{T_x} \). Therefore for every \( f_k \in M_k \), \( f_k \in D_{T_{m/2^n}} \) and therefore belongs to \( D_{T_x} \). Consequently, since \( T_{m/2^n} = H_{m/2^n} \) and by the continuity of \( \langle T_x f_k, f_k \rangle \) and \( \langle H_x f_k, f_k \rangle \) as functions of \( x \), we must have \( T_x f_k = H_x^{(k)} f_k = H_x f_k \). This implies that \( T_x = H_x \) (Sz. Nagy [5, p. 35]).

The theorem for two parameter semi-groups offers no difficulty.

References

5. ———-, Spektraldarstellung linear Transformationen des Hilbertschen Raumes, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 5, 1942.

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