THE ALGEBRA OF BOUNDED FUNCTIONS
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1. Introduction. The Banach algebra $B(X)$ of all bounded complex-valued functions on a set $X$ has been considered by several writers (cf., e.g. [10]). It is our purpose to give an ideal-theoretic characterization of this algebra. The main result (Theorem 3) states necessary and sufficient conditions that a $B^*$-algebra be equivalent to a $B(X)$ on an essentially unique set $X$.

Our method consists in establishing a lattice anti-isomorphism between the lattice of subsets of $X$ and the lattice of all those ideals of $B(X)$ which are defined by annihilation.

These results are also applied to the study of certain $p$-rings.

2. By a $B^*$-algebra is meant a Banach algebra over the complex numbers such that, to each element $x$, there corresponds a unique element $x^*$ (the adjoint of $x$), with the following properties: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$, (iii) if $\alpha, \beta$ are complex numbers, and $\bar{\alpha}, \bar{\beta}$ are their complex conjugates, then $(\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^*$, (iv) $\|x^*x\| = \|x\|^2$.

Let $X$ be any set and denote by $B(X)$ the set of all bounded complex-valued functions defined on $X$. For $f(x) \in B(X)$ define $\|f(x)\| = \sup_{x \in X} |f(x)|$. Under the usual definitions of addition and multiplication and with the norm defined above, and $[f(x)]^* = \overline{f(x)}$, $B(X)$ is a commutative $B^*$-algebra with identity.

If $G$ is any subset of an arbitrary commutative ring $K$, denote by $R(G)$ the set of elements $k$ in $K$ which satisfy $Gk = 0$. $R(G)$ is an ideal of $K$ which we call an annulet. It is closed if $K$ is a topological ring.

$E(X)$ will denote an arbitrary algebra of bounded functions on the set $X$, while the algebra of all functions which are zero at all but a finite set of points of $X$ is denoted by $B_0(X)$.

If $S$ is any subset of $X$, and $G$ a subset of $E(X)$, let $A(S)$ be the set of all $f \in E(X)$ which satisfy $f(S) = 0$, and $N(G)$ the set of all $x \in X$ for which $G(x) = 0$. It is clear that $A(S)$ is a closed ideal of $E(X)$.

If $S, T$ are any subsets of $X$, the logical sum and product are denoted by $S + T$ and $S \cap T$ respectively, while $X - S$ denotes the set of those elements in $X$ which are not in $S$.

**Lemma 1.** If $E(X)$ is an algebra of bounded functions containing the
algebra \( B_0(X) \), then

1. \( R[A(S)] = A(X - S) \) for all \( S \leq X \),
2. \( N[A(S)] = S \) for all \( S \leq X \),
3. \( R(G) = A[X - N(G)] \) for every subset \( G \) of \( E \).

**Proof.** (1) \( A(S)A(X - S) = 0 \) implies that \( A(X - S) \leq R[A(S)] \). If \( S = X \), then \( A(S) = 0 \) so that \( R[A(S)] = E \). If we accept the convention that all functions vanish on the empty set, we see that (1) holds for \( S = X \). Assume \( S < X \) and let \( f \in R[A(S)] \) so that \( A(S)f = 0 \). Let \( x \in X - S \). Then there exists \( g \in E \) satisfying \( g(S) = 0 \) and \( g(*) = 1 \). Thus \( g \in A(S) \) but since \( gf = 0 \) we must have \( f(x) = 0 \). That is, \( f(X - S) = 0 \) or \( f \in A(X - S) \) proving (1).

(2) Since \( A(S)[S] = 0 \) we have \( S \leq N[A(S)] \). To complete the proof we must show \( N[A(S)] \leq S \). We shall assume \( x \in S \) and show \( x \in N[A(S)] \). Write \( X = x + S + U \) and define \( f \) by

\[
    f(S + U) = 0 \quad \text{and} \quad f(x) = 1.
\]

Such an \( f \in E(X) \) by assumption. Then \( f \in A(S) \). But \( f(x) = 1 \) implies \( A(S)(x) \neq 0 \) so that \( x \in N[A(S)] \) which shows (2).

(3) Let \( f \in R(G) \) so that \( Gf = 0 \). Let \( x \in [X - N(G)] \) so that \( x \in N(G) \). This means there exists a function \( g \in G \) such that \( g(x) \neq 0 \). But since \( g(x)f(x) = 0 \) this implies \( f(x) = 0 \). Hence \( f[X - N(G)] = 0 \) and \( f \in A[X - N(G)] \). The reverse implication follows by reversing the steps.

**Theorem 1.** Let \( E(X) \) be an algebra of bounded functions which contains \( B_0(X) \). Then the mapping \( S \to A(S) \) is a duality (lattice anti-isomorphism) between the lattice of all subsets \( S \) of \( X \) and the lattice of all annulets of \( E(X) \).

**Proof.** The mapping \( S \to A(S) \) is one-one by (2) of Lemma 1. Clearly \( S_1 \leq S_2 \) implies \( A(S_1) \leq A(S_2) \) and the mapping is a duality. Every annulet has the form \( A(S) \) by (3) of Lemma 1, and every \( A(S) \) is an annulet by (1) of Lemma 1.

**Theorem 2.** In \( B(X) \) the sum of two annulets is an annulet.

**Proof.** Denote by \( (a) \) the ideal generated by the element \( a \). An ideal of \( B(X) \) is an annulet if and only if it is a principal ideal generated by an idempotent. For, if \( e \) is idempotent \( (e) = R(1 - e) \), and if \( G \) is an annulet, \( G = (e) \) for idempotent \( e \) by Theorem 2.3 of [3]. It is clear that if \( G = A(S) \), the idempotent \( e \) is just the characteristic function of \( X - S \). Now, let the given annulets be \( A(S), A(T) \) where \( A(S) = (e), A(T) = f \) for idempotents \( e \) and \( f \) which satisfy
\[ e(S) = 0, \quad f(T) = 0, \]
\[ e(X - S) = 1, \quad f(X - T) = 1. \]

Since \( A(S) \) and \( A(T) \) are contained in \( A(S \cap T) \) it follows that \( A(S) + A(T) \subseteq A(S \cap T) \). It remains to show that \( A(S \cap T) \subseteq A(S) + A(T) \). Let \( A(S \cap T) = (g) \) where \( g(S \cap T) = 0, \ g\{(X - [S \cap T]\} = 1 \). Write \( X = [S \cap T] + [S - (S \cap T)] + [T - (S \cap T)] + [X - (S + T)] \), a decomposition of \( X \) into four mutually disjoint sets. Then \( g = e + f - ef \) on these four parts separately and hence \( g = e + f - ef \). Hence \( g \subseteq A(S) + A(T) \) and \( (g) \subseteq A(S) + A(T) \), which completes the proof.

**Theorem 3.** A commutative \( B^* \)-algebra \( K \) containing an identity is isomorphic (in a norm and \( * \) preserving manner) to an algebra \( B(X) \) of all bounded complex-valued functions on an essentially unique set \( X \) if and only if:

1. Every nonzero closed ideal of \( K \) contains a minimal ideal.
2. The sum of two annulets is an annulet.

**Proof.** The minimal ideals of \( K \) are of the form \( e_i K \) for \( e_i \) a projection (self-adjoint idempotent) [2, Lemma 4.2]. Each \( e_i K \) is a field and by the Gelfand-Mazur theorem is isomorphic and isometric to the complex field. Consider the mapping \( x \mapsto \{e_i x\} \) for \( x \in K \). Since each \( e_i \) is of norm 1, if we define \( * \) and norm in the obvious way we have a \( * \) homomorphism of \( K \) into the algebra of all bounded sequences. Since \( \prod_i R(e_i K) \) is an intersection of maximal and hence [1, p. 8] closed ideals, it follows from (1) and a theorem of McCoy [5, p. 873], that \( K \) is isomorphic to a special subdirect sum of the \( e_i K \). That is, \( K \) is isomorphic to an algebra \( E(X) \) of bounded functions defined on the set \( X \) of minimal ideals \( e_i K \), where if \( k \in K \), by \( k(e_i K) \) is meant the complex number \( \lambda \) defined by \( e_i k = \lambda e_i \). Moreover, to each \( x \in X \) and complex number \( \lambda \), there exists \( f \in E(X) \) such that \( f(x) = \lambda \), and \( f(X - x) = 0 \). Since \( E(X) \) is a ring we have \( E(X) \supseteq B_0(X) \).

We wish to show that \( E(X) \) contains all characteristic functions of subsets of \( X \). For any subset \( S \subseteq X \), consider the annulets \( A(S), A(X - S) \). By Theorem 1,

\[ A(S) \cup A(X - S) = A(0) = E(X), \]
\[ A(S) \cap A(X - S) = A(X) = 0, \]

where \( \cup \) indicates the lattice join of annulets. But, by (2),

\[ A(S) \cup A(X - S) = A(S) + A(X - S). \]
Hence $E(X) = A(S) \oplus A(X - S)$ in the sense of direct sum of ideals. By a theorem of von Neumann [7, p. 7] it follows, since $E(X)$ contains an identity, that there exists an idempotent $e \leq E(X)$ such that

$$A(S) = (1 - e)E, \quad A(X - S) = eE.$$ 

Since $e \leq A(X - S)$ this implies $e(X - S) = 0$. \(1 - e \in A(S)\) implies \((1 - e)(S) = 0\) or $e(S) = 1$. Hence $E(X)$ contains all characteristic functions and is therefore clearly dense in $B(X)$. Applying Theorem 6.4 of [2] the proof of sufficiency is complete.

By Theorem 1, if $K$ were isomorphic to $B(X_1)$ and $B(X_2)$ there would exist a lattice isomorphism of the lattice of all subsets of $X_1$ upon the lattice of all subsets of $X_2$. In this sense the set $X$ is unique.

The algebra $B(X)$ is certainly a $B^*$-algebra with identity, and satisfies (2) by Theorem 2. Every nonzero ideal of $B(X)$ (not only the closed ones) contains a minimal ideal, as in Theorem 16 of [5] mentioned previously. This completes the proof.

**Remark.** The set $X$ may also be considered to be the set of all maximal annulets of $B(X)$ (those maximal ideals which are annihilators of minimal ideals). Since $B(X)$ may be represented as the algebra $C(M)$ of all continuous functions on the compact space $M$ of its maximal ideals, it is seen that $M$ is the Stone-Čech compactification [9, p. 463] of the discrete space $X$ of maximal annulets. An alternative proof of Theorem 3 could start with the representation $C(M)$.

3. The preceding results with slight modification apply to rings of functions from a discrete set $X$ to a finite field.

In [6] it was shown that a ring of prime characteristic $p$, all of whose elements $x$ satisfy $x^p = x$ (called a $p$-ring), is isomorphic to a subdirect sum of fields $GF(p)$. The ring $P(X)$ of all functions from a set $X$ to $GF(p)$ (the complete direct sum of the fields) is generated by its idempotents, since an arbitrary function in $P(X)$ assumes only a finite number of distinct values and each is an integral multiple of the identity of the field $GF(p)$. Hence we have:

A $p$-ring $K$ containing an identity element is isomorphic to a $P(X)$ if and only if:

1. Every annulet of $K$ contains a minimal ideal.
2. The sum of two annulets is an annulet.

Specializing this to Boolean rings ($p = 2$) we get:

A Boolean ring $K$ containing an identity is isomorphic to the ring of all subsets of a set $S$ if and only if it satisfies conditions (1) and (2) above.
This result is similar to one obtained by Stone [8, p. 98, Theorem 62].

The complete direct sum of the fields $GF(p^n)$ is generated by its idempotents when considered as an algebra over $GF(p^n)$ (although not as a ring). Hence we have:

Let $K$ be an algebra (containing an identity element) over the Galois field $GF(p^n)$, all of whose elements satisfy the equation $x^{p^n} = x$. Then $K$ is isomorphic to the algebra $W(X)$ of all functions from a set $X$ to the field $GF(p^n)$ if and only if $K$ satisfies the conditions (1) and (2) (cf. [4, p. 379, Theorem 3]).

References


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