ON THE CHARACTERISTIC POLYNOMIAL OF THE
PRODUCT OF TWO MATRICES

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The following theorem will be proved:

**Theorem.** If $A$ and $B$ are $n \times n$ matrices with elements in the field $F$, whose characteristic polynomials are

$$a_0(x^2) - xa_1(x^2) \text{ and } b_0(x^2) - xb_1(x^2)$$

respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$ and $b_1(x)$ are elements in the polynomial domain, $F[x]$, of $F$; and if the rank of $A - B$ does not exceed unity; then the characteristic polynomial of $AB$ is $(-1)^n[a_0(x)b_0(x) - xa_1(x)b_1(x)]$.

The proof will be facilitated by two lemmas.

**Lemma I.** If the rank of an $n \times n$ matrix, $D$, with elements in $F$ does not exceed unity, then there exist 1 $\times$ $n$ matrices $R$ and $S$ with elements $r_i$ and $s_i$, $i=1, 2, \cdots, n$, in $F$ such that $D=RTS$, where $RT$ is the transpose of $R$.

**Lemma II.** If $M=(m_{ij})$ is an $n \times n$ matrix with elements in $F[x]$ and if $D$ is an $n \times n$ matrix as defined by Lemma I, then the determinant of $M+D$ is given by

$$| M + D | = | M | + SM^AR^T,$$

where $M^A$ is the adjoint of the matrix $M$ and $D=RTS$.

The validity of Lemma I is obvious. The rank of $D$ does not exceed unity, hence every two of its rows (columns) are proportional and $D=(r_is_j)=RTS$, where $R=(r_1, r_2, \cdots, r_n)$ and $S=(s_1, s_2, \cdots, s_n)$ and $r_i$ and $s_i$, $i=1, 2, \cdots, n$, are in $F$.

To prove Lemma II, let $D=RTS$, where $R$ and $S$ are matrices established by Lemma I. The determinant of $M+D$ may be expressed as the sum of $2^n$ determinants. Of these $|M|$ is one; $n$ others are $|M_i|$, $i=1, 2, \cdots, n$, where the matrix $M_i$ is obtained from $M$ by replacing its $i$th row by $Sr_i$ of $D$; and the remaining $2^n-1-n$ are zeros for their matrices are obtained by replacing two or more of the

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rows of $M$ by corresponding rows of $D$. Upon expanding $|M_i|$ in terms of its $i$th row, we find

$$|M_i| = \left( \sum_{j=1}^{n} s_j M_{ji} \right) r_i,$$

where $M_{ji}$, $i, j = 1, 2, \ldots, n$, is the cofactor of $m_{ij}$ of $M$. Consequently

$$|M - D| = |M| + \sum_{i=1}^{n} |M_i| = |M| + SM^A R^T,$$

where $M^A$ is the adjoint of $M$. The lemma is proved.

We now take up the main theorem. By hypotheses

$$|xI - A| = a_0(x^2) - xa_1(x^2), \quad |xI - B| = b_0(x^2) - xb_1(x^2).$$

Let $(xI - A)^A = A_0(x^2) - xA_1(x^2)$, where $A_0(x^2)$ and $A_1(x^2)$ are polynomials in $A$ and $x^2I$ and are therefore commutative with $A$. Then

$$(xI - A) \left[ A_0(x^2) - xA_1(x^2) \right] = [a_0(x^2) - xa_1(x^2)] I,$$

and consequently

$$(2) \quad a_0 I = -AA_0 - x^2 A_1, \quad a_1 I = -AA_1 - A_0. \quad (1)$$

Since the rank of $A - B$ does not exceed unity, we may, according to Lemma I, let $A - B = D = R^T S$ and as a result of Lemma II

$$|xI - B| = |xI - A + D| = |xI - A| + S(xI - A)^A R^T = a_0 - xa_1 + S(A_0 - xA_1) R^T,$$

and by equation (1) it follows that

$$(3) \quad b_0 = a_0 + SA_0 R^T, \quad b_1 = a_1 + SA_1 R^T.$$

According to (2) and (3) and because $A$, $A_0$, and $A_1$ are commutative, we find that

$$a_0 b_0 - x a_1 b_1 = a_0(a_0 + SA_0 R^T) - x a_1(a_1 + SA_1 R^T),$$

$$= a_0 - x a_1 + S(a_0 A_0 - x a_1 A_1) R^T,$$

$$= a_0 - x a_1 - S[(AA_0 + x^2 A_1) A_0 - x^2 (AA_1 + A_0 A_1) R^T],$$

$$= a_0 - x a_1 - S(A_0 - x^2 A_1) R^T. \quad (4)$$

1 Here and in the following discussion we suppress the argument $x^2$ of polynomials until the final step in the proof.
Moreover since $B = A - D$

$$x^2I - AB = x^2I - A^2 + AD = x^2I - A^2 + A R^T S.$$ 

The rank of $AD = (A R^T) S$ does not exceed that of $D$, consequently by Lemma II we conclude that

$$|x^2I - AB| = |x^2I - A^2| + S(x^2I - A^2) A R^T.$$ 

It can be shown that

$$|x^2I - A^2| = (-1)^n (a_0^2 - x^2 a_1^2),$$

$$(x^2I - A^2)^A = (-1)^{n-1} (A_0^2 - x^2 A_1^2).$$

Hence we have

$$|x^2I - AB| = (-1)^n [a_0^2 - x^2 a_1^2 - S(A_0^2 - x^2 A_1^2) A R^T]$$

$$= (-1)^n (a_0 b_0 - x^2 a_1 b_1),$$

according to equation (4). If in this equation we replace $x^2$ by $x$, the proof of the theorem is complete.

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