

ON SOME CHARACTERISTIC PROPERTIES OF QUASI-FROBENIUS AND REGULAR RINGS

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Recently the first writer [1] gave a characterization of quasi-Frobenius rings, introduced formerly by the second writer [3], in terms of a condition proposed by K. Shoda, which reads: A ring A satisfying minimum condition and possessing a unit element is a quasi-Frobenius ring if and only if A satisfies the following condition:¹

(a) every (A -left-) homomorphism of a left-ideal of A into A may be given by the right multiplication of an element of A .

In the present note we shall offer a simpler² proof of this, making use of the second writer's former characterization of quasi-Frobenius rings and a theorem in a previous joint note of the writers. The present approach starts, contrary to the one in [1],³ with a theorem (Theorem 1) which is independent of any chain condition (or is concerned with maximum condition at most (corollary to Theorem 1) and which is perhaps of interest by itself. We shall also show that a remark at the end of [1], concerning semisimple rings with minimum condition, may be freed from chain condition, to yield a certain characterization of von Neumann's regular rings (Theorem 3).

1. **Conditions (a), (a*).** Let A be an (associative) ring. On weakening (a), we consider the condition:

(a*) every homomorphism of a finitely generated left-ideal of A into A may be given by the right multiplication of an element of A .

On restricting ourselves further to principal left-ideals, we obtain a (further weaker) condition, which we denote by (a**).

On the other hand, with a subset X in A , we denote, as in our former papers, the set of left (right) annihilators of X in A by $l(X)$ ($r(X)$). We consider the conditions⁴

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¹ Under the assumption of the existence of unit element, this is equivalent to Shoda's condition that every homomorphism between two left-ideals should be extended to an endomorphism of A (considered as A -left-module).

² Simpler, provided we used the results in our former works.

³ On the other hand, the proof in [1] gives some further details for the case with minimum condition.

⁴ Observe that the dual relations $r(l_1 + l_2) = r(l_1) \cap r(l_2)$ is trivial (and is always true). Contrary to our former papers, we use "+" for a mere module sum which is not necessarily direct.

$$(b) \quad r(I_1 \cap I_2) = r(I_1) + r(I_2),$$

$$(c) \quad r(l(r)) = r,$$

concerning, respectively, pairs of left-ideals I_1, I_2 of A and right-ideals r of A . We denote by (b^*) , (c^*) the conditions which demand the validity of (b), (c), respectively, for finitely generated left- or right-ideals. On restricting (c) further to principal right-ideals we introduce the condition (c^{**}) .

THEOREM 1. *Let A be a ring with unit element. Then the following implications hold in A :*

- (i) $(a^{**}) \leftrightarrow (c^{**})$;
- (ii) $(a^*) \leftrightarrow (b^*)$, (c^*) ;
- (iii) $(a) \rightarrow (b)$, (c^*) .

PROOF.⁵ (i) Assume (a^{**}) . We consider a principal left-ideal Aa . It is (A -left)-isomorphic to the residue-module $A/l(a)$ by the correspondence $x(\text{mod } l(a)) \leftrightarrow xa$. Let b be an arbitrary element in $r(l(a))$. As $l(b) \supseteq l(a)$, the mapping $xa \rightarrow xb$ gives a homomorphism of Aa onto Ab . According to (a^{**}) there must exist an element c in A satisfying $ac = b$, and hence $b \in aA$. This shows $r(l(aA)) (=r(l(a)))$ (\subseteq whence) $= aA$.

Assume conversely (c^{**}) . Let Aa be a principal left-ideal in A and let θ be a homomorphism of Aa into A . Clearly $l(a) \subseteq l(a^\theta)$. Hence, by (c^{**}) , $a^\theta A = r(l(a^\theta A)) \subseteq r(l(aA)) = aA$, and there exists an element c with $a^\theta = ac$. The right-multiplication of c onto Aa gives the homomorphism θ .

(ii) Assume (a^*) . Let I_1, I_2 be two finitely generated left-ideals and let b be an arbitrary element in $r(I_1 \cap I_2)$. Taking an element c of A arbitrarily,⁶ we consider the homomorphic mappings

$$\begin{aligned} \theta_1: \quad x_1 &\rightarrow x_1c & (x_1 \in I_1), \\ \theta_2: \quad x_2 &\rightarrow x_2(c + b) & (x_2 \in I_2) \end{aligned}$$

of I_1 and I_2 onto I_1c and $I_2(c+b)$, respectively. As $b \in r(I_1 \cap I_2)$, these mappings θ_1, θ_2 coincide on $I_1 \cap I_2$ and together generate a homomorphic mapping θ of $I_1 + I_2$ onto $I_1c + I_2(c+b)$:

$$x_1 + x_2 \rightarrow x_1c + x_2(c + b).$$

By (a^*) this homomorphism θ must be given by the right-multiplication of an element, say a , and we must have thus

⁵ A half of our proof of (i), (ii) is a mere reproduction of the arguments in [1]. The other half is to follow them in the reverse order.

⁶ We may take for instance the unit element for c .

$$c - a \in r(I_1), \quad (c + b) - a \in r(I_2).$$

Here $b = (c + b - a) - (c - a) \in r(I_1) + r(I_2)$. This shows that $r(I_1 \cap I_2)$ (\subseteq whence) $= r(I_1) + r(I_2)$, proving (b*). That (c**) holds has been secured in (i).

Assume conversely (b*) and (c**). Then (a**) holds as has been seen in (i). Let $I = Aa_1 + Aa_2 + \cdots + Aa_n$ be any finitely generated left-ideal in A , and suppose that (a*) is the case for its left-subideal $I_1 = Aa_1 + Aa_2 + \cdots + Aa_{n-1}$. Let θ be a homomorphism of I into A , and let θ_1, θ_2 be the restrictions of θ to I_1, Aa_n respectively. Each of θ_1, θ_2 is given by the right-multiplication of an element of A , say c_1 or c_2 respectively. As θ_1 and θ_2 coincide on $I_1 \cap Aa_n$, we have $c_1 - c_2 \in r(I_1 \cap Aa_n)$. The last right-ideal is, by (b*), equal to $r(I_1) + r(Aa_n)$. Thus $c_1 - c_2 = b_1 - b_2$ with $b_1 \in r(I_1), b_2 \in r(Aa_n)$. Denote $c_1 + b_1 (= c_2 + b_2)$ by c . Then the right-multiplication of c onto $I (= I_1 + Aa_n)$ gives our homomorphism θ . Now (a*) can generally be established by induction.

(iii) Assume (a). On repeating our consideration in the first half of (ii), this time with arbitrary left-ideals I_1, I_2 , we see the validity of (b). Combining that with (c**) (already established in (i)) we have

$$\begin{aligned} r(l(a_1A + a_2A + \cdots + a_nA)) \\ &= r(l(a_1A) \cap l(a_2A) \cap \cdots \cap l(a_nA)) \\ &= r(l(a_1A)) + r(l(a_2A)) + \cdots + r(l(a_nA)) \\ &= a_1A + a_2A + \cdots + a_nA \end{aligned}$$

for any finitely generated right-ideal $r = a_1A + a_2A + \cdots + a_nA$, proving (c*).

COROLLARY. *Let A be a ring with unit element and satisfying maximum condition. Then A satisfies (a) (equivalently (a*)) if and only if the correspondence $I \rightarrow r(I)$ (I : left-ideal of A) gives a (lattice-theoretical) dual-homomorphism of the left-ideal lattice of A onto the right-ideal lattice of A .*

PROOF. In our A the conditions (a), (a*) are equivalent and (b), (b*) are equivalent. (ii) and (iii) of Theorem 1 together give our corollary.

2. Quasi-Frobenius rings. Let now A be a ring with unit element satisfying minimum condition (whence maximum condition too). Our Theorem 1 (or its corollary), combined with Theorem 6 of [3], shows that A satisfies (a) if it is quasi-Frobeniusean. To prove the converse, assume that A satisfies (a). Then, by Theorem 1 (or its corol-

lary), we have (c) in A . Hence, according to Theorem 1 of [2]⁷; there exists for any maximal left-ideal I_1 in A a minimal left-ideal I_2 in A with $A/I_1 \cong I_2$. Let now I be an arbitrary left-ideal in A . We want to show that $I = l(r(I))$. To do so, assume the contrary (i.e. $I \subsetneq l(r(I))$) and let I_0 be a left-subideal of $l(r(I))$ in which I is maximal. There exists (a maximal left-ideal I_1 of A such that $I_0/I_1 \cong A/I_1$, whence there exists) a minimal left-ideal I_3 of A isomorphic to I_0/I . Thus we have a homomorphism of I_0 onto I_3 , mapping I onto 0, which must be, by (a), given by the right-multiplication of an element, say c . We have $Ic = 0$, i.e. $c \in r(I)$, whence $I_0 \subseteq l(r(I)) \subseteq l(c)$, $I_0c = 0$, contrary to $I_0c = I_3$. Thus necessarily $I = l(r(I))$ for any left-ideal I in A . This and (c) show, again by Theorem 6 of [3], that A is quasi-Frobeniusean. Thus we have

THEOREM 2. *A ring A with unit element and satisfying minimum condition is quasi-Frobeniusean if and only if it satisfies (a).*

3. Regular rings.

THEOREM 3. *A ring A with unit element is regular if and only if it satisfies the following condition:*

(a_0^*) *every (A -left-) homomorphism of a principal (or finitely generated) left-ideal I of A into (or onto) a residue-module A/I' , of A modulo a left-ideal I' , may be obtained by the right-multiplication of an element, say c , of A : $x \rightarrow xc \pmod{I'}$ ($x \in I$).*

PROOF. Suppose first that (a_0^*) is the case. With an arbitrary element a in A , set $I' = l(a)$. The isomorphism $ya \rightarrow y \pmod{I'}$ ($y \in A$) of Aa onto A/I' must be obtained by the right-multiplication of an element, say c , of A . Thus $y \equiv yac \pmod{I'}$, whence $ya = yaca$, for any element y in A . So $a = aca$, and A is a regular ring.

Assume conversely that A is a regular ring. (Any finitely generated right-ideal is principal and) any principal right-ideal aA is generated by an idempotent element, say e . Let $e = ab$. We have $y \equiv ye = yab \pmod{l(a)} (= l(aA) = l(e))$ for any $y \in A$. If we have a homomorphism of Aa into a residue-module A/I' and if the image of a is $a_1 \pmod{I'}$, then we ought to have $l(a)a_1 \subseteq I'$ and $(y - yab)a_1 \subseteq I'$, i.e., $ya_1 \equiv yaba_1 \pmod{I'}$ for any $y \in A$. Thus our homomorphism of Aa into A/I' is given by $ya \rightarrow yac$ ($y \in A$) with $c = ba_1$, which proves (a_0^*).

REFERENCES

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⁷ To use this theorem is more or less equivalent to using Lemma 2 in [1].

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THE NUMBER OF SUBGROUPS OF GIVEN INDEX IN NONDENUMERABLE ABELIAN GROUPS

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Let G be an Abelian group of order $A > \aleph_0$. It has been shown [4, Theorem 9] that there exist 2^A subgroups of G of order A , and that the intersection of all such subgroups is 0. In this paper, this result is improved to the following: If $\aleph_0 \leq B \leq A$ and $A > \aleph_0$, then an Abelian group of order A has 2^A subgroups of index B , and the intersection of all such subgroups is 0. In addition, it is shown that there is a set of 2^A subgroups H_α of index B such that $G/H_\alpha \cong G/H_{\alpha'}$ for all α, α' .

Baer [1, p. 124] showed that if G is an Abelian p -group which is the direct sum of A cyclic groups of bounded order, then G has 2^A subgroups of index p (here A may equal \aleph_0). The proof in the present paper is accomplished by extending Baer's result in an obvious manner to a wider class of p -groups, and then reducing all other cases to this one.

We shall use $+$ and \sum to denote direct sums, and $o(S)$ to denote the number of elements in S .

LEMMA. *Let $H \neq 0$ be an Abelian group, and let $G = \sum H_\alpha$, $\alpha \in S$, $H_\alpha \cong H$ for all $\alpha \in S$, $o(S) = A \geq \aleph_0$. Then there are at least 2^A subgroups K_β of G such that $G/K_\beta \cong H$.*

PROOF. (This proof is the same as Baer's, and is included only for the sake of completeness.) Identify H_α with H . Let ϵ_α be 0 or 1 for each $\alpha, \alpha \in S$. Let K be the set of elements of G such that $h_{\alpha_0} = \sum \epsilon_\alpha h_\alpha$, $\alpha \neq \alpha_0$. Then it is easy to verify that K is a subgroup of G and $G = H_{\alpha_0} + K$. If $\epsilon_\alpha = 0$, then $H_\alpha \subset K$, but if $\epsilon_\alpha = 1$, then $H_\alpha \cap K = 0$. Thus all of the K 's are distinct, and the lemma is proved.

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