REGIONS OF EXCLUSION FOR THE LATENT ROOTS OF A MATRIX

HANS SCHNEIDER

The well-known Theorem 0 is due to S. Gersgorin [2] and A. Brauer [1]:

**Theorem 0.** Let \( A: \begin{bmatrix} a_{ij} \end{bmatrix} \) be an \( n \times n \) matrix with complex elements. Then the latent roots of \( A \) lie in the union of the \( n \) circular regions \( |z - a_{ii}| \leq P_i, i = 1, \ldots, n, \) in the complex plane, where \( P_i = \sum_{j=1}^{j=n} |a_{ij}| \) and \( \sum' \) denotes summation from \( j = 1 \) to \( j = n \) with \( j = i \) omitted.

It is the purpose of this note to point out that there may also exist bounded regions which exclude the latent roots of \( A \).

Let \( B: \begin{bmatrix} b_{ij} \end{bmatrix} \) be an \( n \times n \) matrix with complex elements. Let \((\mu(1), \ldots, \mu(n))\) be a permutation of \((1, \ldots, n)\), and let \( B': \begin{bmatrix} b'_{ij} \end{bmatrix} \) be the matrix \( \begin{bmatrix} b'_{ij} \end{bmatrix} = b_{\mu(j)i} \). Thus \( B' \) is obtained from \( B \) by a permutation of its columns, and therefore \( B \) is nonsingular when \( B' \) is nonsingular.

The matrix \( B' \) is nonsingular if

\[
| b'_{ii} | > \sum' | b'_{ij} | , \quad \text{for } i = 1, \ldots, n ,
\]

(cf. e.g. O. Taussky [5]), and so \( B \) is nonsingular when

\[
(1) \quad | b_{\mu(i)i} | > \sum' | b_{\mu(j)i} | = \sum_{j \neq \mu(i)} | b_{ij} | , \quad \text{for } i = 1, \ldots, n .
\]

Now let \( B = \lambda I - A \), where \( \lambda \) is a latent root of \( A \). Then \( b_{ii} = \lambda - a_{ii} \), and \( b_{ij} = -a_{ij} \), when \( i \neq j \). Since \( B \) is singular not all the inequalities (1) can hold. Hence there is an \( i, 1 \leq i \leq n \), such that either

\[
| \lambda - a_{ii} | \leq P_i \quad \text{and} \quad i = \mu(i) ,
\]

or

\[
| a_{\mu(i)i} | \leq | \lambda - a_{ii} | + \sum'' | a_{ij} | \quad \text{and} \quad i \neq \mu(i) ,
\]

where \( \sum'' \) denotes summation from \( j = 1 \) to \( j = n \), with \( j = i \) and \( j = \mu(i) \) omitted. We now immediately obtain Theorem 1.

**Theorem 1.** Let \( A \) be an \( n \times n \) matrix with complex elements. Let \((\mu(1), \ldots, \mu(n))\) be a permutation of \((1, \ldots, n)\). Then the latent roots of \( A \) lie in the union of the \( n \) regions

Received by the editors July 21, 1953.
|z - a_{ii}| \leq P_i \quad \text{when} \quad i = \mu(i),
|z - a_{ii}| \geq Q_i \quad \text{when} \quad i \neq \mu(i),

where \(Q_i = |a_{\mu(i)}| - \sum_{i' \neq i} |a_{ii'}| \).

If \(\mu(i) = i\) when \(i = 1, \ldots, n\) then Theorem 1 reduces to Theorem 0. In this case we obtain a bounded region in the complex plane within which all latent roots of \(A\) lie. If some \(i \neq \mu(i)\), Theorem 1 may yield a bounded region of the complex plane within which no latent root of \(A\) can lie. Let \(I\) be the union of the interiors and boundaries of the circles \(|z - a_{ii}| = P_i\) when \(i = \mu(i)\). Let \(CI\) be the complement of \(I\). Let \(E\) be the intersection of the interiors of the circles \(|z - a_{ii}| = Q_i\) when \(i \neq \mu(i)\), such an interior being empty when \(Q_i \leq 0\). It is easily seen that the region of exclusion for the latent roots of \(A\) given by Theorem 1 is the intersection of \(CI\) and \(E\). If \(E\) is empty, or if \(E\) is contained in \(I\), there will be no region of exclusion. In particular, there is no region of exclusion if \(Q_i \leq 0\) for some \(i \neq \mu(i)\).

We shall apply Theorem 1 to the matrix

\[
A = \begin{bmatrix}
1 & 4 & 1 & 0 \\
4 & 2 & 0 & 0 \\
0 & 1 & -1 & 2 \\
1 & 0 & 3 & -2
\end{bmatrix}.
\]

(In the case of nondiagonal elements only the absolute values are relevant.) From the permutation \((1, 2, 3, 4)\) it follows by means of Theorem 1 (or Theorem 0) that the latent roots of \(A\) lie within or on the circles \(|z - 1| = 5\) or \(|z + 2| = 4\). From the permutation \((2, 1, 3, 4)\) it follows that no latent root of \(A\) lies in that part of the interior of the circle \(|z - 1| = 3\) which is outside \(|z + 2| = 4\). From the permutation \((2, 1, 4, 3)\) it follows that there is no latent root of \(A\) within \(|z + 1| = 1\). No region of exclusion is obtained from \((1, 2, 4, 3)\) even though \(Q_3\) and \(Q_4\) are positive.

If \(n = 2\) the only possible permutations of \((1, 2)\) are \((1, 2)\) and \((2, 1)\). Let \(\lambda\) be a latent root of the \(2 \times 2\) matrix \(A\). Applying Theorem 1 with the permutation \((1, 2)\) we obtain that \(\lambda\) lies within or on one of the circles

\[
|z - a_{11}| = |a_{12}|, \quad |z - a_{22}| = |a_{21}|.
\]

When Theorem 1 is used with the permutation \((2, 1)\) it follows that \(\lambda\) lies outside or on one of the circles \((2)\). Thus \(\lambda\) lies inside or on one of the circles \((2)\), but not inside both. This is a slightly weakened form of a theorem due to O. Taussky [6].
Some other known results may be generalized in the same way as Theorem 0. P. Stein has proved that if $B'$ is singular and its latent root $0$ has $m$ associated linearly independent latent column vectors, then at least $m$ of the inequalities

$$|b'_{ii}| \leq \sum_{j}^{} |b'_{ij}|,$$

$i = 1, \ldots, n,$

are satisfied (P. Stein [4], O. Taussky [7]).

The latent column vectors of $B'$ associated with $0$ are just those of $B$ after the permutation $(\mu(1), \cdots, \mu(n))$ has been applied to the indices of the elements of the latter. Hence $B$ has precisely as many linearly independent latent column vectors associated with $0$ as $B'$. If we now apply the argument leading to Theorem 1 to Stein's result we obtain Theorem 2.

**Theorem 2.** Let $\lambda$ be a latent root of $A$ which has $m$ linearly independent latent column vectors associated with it. Let $(\mu(1), \cdots, \mu(n))$ be a permutation of $(1, \cdots, n)$. Then $\lambda$ lies in at least $m$ of the regions

$$|z - a_{ii}| \leq P_i, \text{ when } i = \mu(i),$$

$$|z - a_{ii}| \geq Q_i, \text{ when } i \neq \mu(i).$$

Results which apply to irreducible matrices only do not seem to be extendable in this simple manner. The reason is that $B'$ may be reducible when $A$ and $B = \lambda I - A$ are irreducible. It is possible to apply the above method to more general bounds for the latent roots of a matrix, such as those of Ostrowski [3], but the regions of exclusion obtained can apparently not be expressed in a simple form.

**References**


The Queen's University, Belfast