PROOF OF THE WELL-ORDERING OF CARDINAL NUMBERS

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It is well known that the class of cardinal numbers is well-ordered. But the proofs that we know are long ones, using Zorn's Theorem and the cumbersome theory of ordinal numbers. In this paper we give a very short proof of this theorem using both the Axiom of Choice and Zorn's Theorem.

By the theorem of Bernstein-Cantor we know that the cardinal numbers form an order class. If we prove that every family of cardinal numbers has a first element it will follow that it is totally ordered (if not, a set of two incomparable elements would not have a first element) and indeed, well-ordered.

We shall use the notations and terminology of Bourbaki.

Theorem 1. Let \((\mathcal{N}_i)_{i \in I}\) be a family of cardinal numbers of subsets \(A_i\) of a set \(E\). Then there exists an \(i_0 \in I\) such that \(\mathcal{N}_{i_0} \leq \mathcal{N}_i\) for every \(i \in I\).

Proof. We shall have our result if we prove that for every \(i \in I\) there exists a one-to-one function \(f_i\) from \(A_{i_0}\) into \(A_i\).

Let us take the cartesian product \(A = \prod_{i \in I} A_i\) and let us consider the class \(\mathcal{B}\) of subsets \(B\) of \(A\) such that

\[ (*) \quad x = (x_i) \in B, \quad y = (y_i) \in B, \quad \text{and} \quad x \neq y \implies x_i \neq y_i, \quad \text{for every} \quad i \in I. \]

It is immediate that \(\mathcal{B}\) is inductive when ordered by inclusion and hence, by Zorn's Theorem, \(\mathcal{B}\) has at least one maximal element \(\overline{B}\). We assert that there exists an \(i_0 \in I\) such that \(pr_{i_0}(\overline{B}) = A_{i_0}\): for if \(pr_i(\overline{B}) \neq A_i\) for every \(i \in I\) we could take (by the Axiom of Choice) an element \(a_i\) in every \(A_i - pr_i(\overline{B})\) and set \(\overline{B} \cup \{a\}\) where \(a = (a_i)\) would strictly contain \(\overline{B}\) and it would still satisfy (*) and so \(\overline{B}\) would not be maximal. If \(pr_{i_0}(\overline{B}) = A_{i_0}\) we define the one-to-one function \(f_i\) from \(A_{i_0}\) into \(A_i\) by \(x_{i_0} \in A_{i_0} \rightarrow f_i(x_{i_0}) = x_i = pr_i x\) where \(x\) is the point of \(\overline{B}\) such that \(pr_{i_0}x = x_{i_0}\); by (*) this point \(x \in \overline{B}\) is unique and \(f_i\) is one-to-one \((x_{i_0} \neq y_{i_0} \rightarrow x \neq y \rightarrow x_i \neq y_i)\). Q.E.D.

Corollary. The cardinal numbers of subsets of a set \(E\) form a well-ordered set.

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