REMARK ON A CERTAIN CLASS OF CONTINUED FRACTIONS

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1. **Introduction.** Continued fractions of the form

$$(1.1)\quad k_0\gamma_0 + \frac{k_0(1-\gamma_0\bar{\gamma}_0)z}{\bar{\gamma}_0 z} \;\underset{-}{}\; \frac{1}{k_1\gamma_1} + \frac{k_1(1-\gamma_1\bar{\gamma}_1)z}{\bar{\gamma}_1 z} \;\underset{-}{}\; \frac{1}{k_2\gamma_2} + \cdots,$$

where the k_p and γ_p are arbitrary constants such that $k_p \neq 0$, $|\gamma_p| \neq 1$, $\gamma_p \neq 0$, $p=0, 1, \cdots$, were considered by Frank [1].[1] It was shown that to every such continued fraction there corresponds a unique power series $c_0+c_1z+c_2z^2+\cdots$ such that the Taylor's series for the approximants $A_{2p}(z)/B_{2p}(z)$, $A_{2p+1}(z)/B_{2p+1}(z)$ agree with the power series up to and including the terms involving z^p and z^{p-1}, respectively. An algorithm was also given for the computation of the γ_p when the power series and the k_p are given. In particular, for the geometric series

$$(1.2)\qquad 1 + z + z^2 + z^3 + \cdots$$

and $k_p=2$, it was shown that $\gamma_p=(-1)^p/(p+2)$, and the corresponding continued fraction was found to be

$$(1.3)\quad 1 + \frac{2\cdot\frac{3}{4}z}{\frac{1}{2}z} \;\underset{-}{}\; \frac{1}{-\frac{2}{3}} + \frac{2\cdot\frac{8}{9}z}{-\frac{1}{3}z} \;\underset{-}{}\; \frac{1}{\frac{2}{4}} + \frac{2\cdot\frac{15}{16}z}{\frac{1}{4}z} \;\underset{-}{}\; \frac{1}{-\frac{2}{5}} + \frac{2\cdot\frac{24}{25}z}{-\frac{1}{5}z} - \cdots.$$

In this paper the function to which the expansion (1.1) corresponding to the geometric series (1.2) converges is considered. In §2 the general formulas for such a continued fraction are given. Then, by examples, it is shown that several cases occur: (i) (1.1) converges over the entire z-plane[2] and equals $1/(1-z)$ (which is the function one would expect to obtain) (Theorem 3.1), or equals an *entirely different function* (Theorem 3.2); (ii) (1.1) converges in different regions to entirely different functions (§4). In this case it is shown that the continued fraction (1.3) converges to $1/(1-z)$ in $|z|<1/2$ and to $4-1/z$ in $|z|>1/2$.

2. **The continued fraction (1.1) corresponding to the geometric**

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Here the origin is always excluded since the B_{2p+1} are zero for $z=0$ and consequently the odd approximants are indeterminate.

series $1+z+z^2+\cdots$. The recurrence formulas for the numerators of the approximants of (1.1) are[3]

$$
\begin{aligned}
A_0 &= k_0\gamma_0 = 1, \qquad A_1 = k_0 z, \\
A_{2p-2} &= k_{p-1}\gamma_{p-1}A_{2p-3} - A_{2p-4}, \\
A_{2p-1} &= \bar{\gamma}_{p-1}zA_{2p-2} + k_{p-1}(1-\gamma_{p-1}\bar{\gamma}_{p-1})zA_{2p-3}, \\
A_{2p} &= k_p\gamma_p A_{2p-1} - A_{2p-2}, \qquad p = 2, 3, \cdots .
\end{aligned}
\tag{2.1}
$$

By (3.4) of [1],

$$
\begin{aligned}
&(1+z+z^2+\cdots)B_{2p} - A_{2p} \\
&\quad = -k_{p+1}\gamma_{p+1}z^{p+1}\prod_{i=0}^{p} k_i(1-\gamma_i\bar{\gamma}_i) + \cdots, \qquad p = 0, 1, \cdots .
\end{aligned}
$$

By writing $1+z+z^2\cdots = 1/(1-z)$, one obtains

$$
B_{2p} - (1-z)A_{2p} = -k_{p+1}\gamma_{p+1}z^{p+1}\prod_{i=0}^{p} k_i(1-\gamma_i\bar{\gamma}_i).
$$

Since $A_{2p} = d_p z^p + d_{p-1}z^{p-1} + \cdots$, by equating the coefficients of z^{p+1}, one finds from (2.1) that

$$
d_p = \gamma_p\prod_{i=0}^{p} k_i = -k_{p+1}\gamma_{p+1}\prod_{i=0}^{p} k_i(1-\gamma_i\bar{\gamma}_i)
$$

or

$$
k_{p+1}\gamma_{p+1} = \frac{-\gamma_p}{\prod_{i=0}^{p}(1-\gamma_i\bar{\gamma}_i)}, \qquad p = 0, 1, \cdots . \tag{2.2}
$$

By division of successive terms in (2.2), one obtains

$$
k_{p+1}\gamma_{p+1} = \frac{k_p\gamma_p^2}{\gamma_{p-1}(1-\gamma_p\bar{\gamma}_p)}, \quad (\gamma_{-1} = -1), \qquad p = 0, 1, \cdots . \tag{2.3}
$$

Furthermore, by elimination of A_{2p-1} and A_{2p-3} in (2.1), it follows that

$$
A_{2p} - \frac{k_p\gamma_p}{\gamma_{p-1}}zA_{2p-2} = -\left[A_{2p-2} - \frac{k_p\gamma_p(1-\gamma_{p-1}\bar{\gamma}_{p-1})}{\gamma_{p-1}}zA_{2p-4}\right],
$$

or, by (2.3),

[3] Instead of $A_p(z)$, $B_p(z)$, the notation A_p, B_p will be used throughout.

$$(2.4)\qquad A_{2p} - \frac{k_p\gamma_p}{\gamma_{p-1}} zA_{2p-2} = -\left[A_{2p-2} - \frac{k_{p-1}\gamma_{p-1}}{\gamma_{p-2}} zA_{2p-4}\right],$$
$$p = 2, 3, \cdots.$$

By substitution of successive values of p in (2.4), one has the recurrence relation

$$A_{2p} - \frac{k_p\gamma_p}{\gamma_{p-1}} zA_{2p-2} = (-1)^{p-1}\left[A_2 - \frac{k_1\gamma_1}{\gamma_0} zA_0\right] = (-1)^{p-1}P,$$
$$p = 1, 2, \cdots.$$

If one displays these equations for $p=1, 2, \cdots$, and multiplies them with the proper factors so that by addition the terms involving $A_{2p-2}, A_{2p-4}, \cdots, A_2$ are eliminated on the left-hand side, one obtains the following relations with series in increasing powers of z,

$$(2.5)\qquad \begin{aligned} A_{2p} &= \gamma_p k_p k_{p-1} \cdots k_1 z^p \frac{A_0}{\gamma_0} \\ &+ P\cdot\left[(-1)^{p-1} + (-1)^{p-2}\frac{\gamma_p k_p}{\gamma_{p-1}} z + \cdots \right. \\ &\left. - \frac{\gamma_p k_p k_{p-1} \cdots k_3}{\gamma_2} z^{p-2} + \frac{\gamma_p k_p k_{p-1} \cdots k_2}{\gamma_1} z^{p-1}\right], \end{aligned}$$

or with series in decreasing powers of z,

$$(2.6)\qquad \begin{aligned} \frac{\gamma_0 A_{2p}}{\gamma_p k_1 k_2 \cdots k_p z^p} &= A_0 + P\cdot\left[\frac{\gamma_0}{k_1\gamma_1 z} - \frac{\gamma_0}{k_1 k_2 \gamma_2 z^2} + \cdots \right. \\ &\left. + (-1)^{p-1}\frac{\gamma_0}{k_1 k_2 \cdots k_p \gamma_p z^p}\right], \qquad p = 1, 2, \cdots. \end{aligned}$$

Analogous formulas hold for the denominators B_{2p}.

Similar relations are obtained for the A_{2p-1} and B_{2p-1} from the last formula of (2.1),

$$(2.7)\qquad k_p\gamma_p A_{2p-1} = A_{2p} + A_{2p-2}, \quad k_p\gamma_p B_{2p-1} = B_{2p} + B_{2p-2}.$$

Furthermore,

$$(2.8)\qquad \begin{aligned} A_0 &= k_0\gamma_0 = 1, & P &= A_2 - \frac{k_1\gamma_1}{\gamma_0} zA_0 = -k_0\gamma_0 = -1; \\ B_0 &= 1, & Q &= B_2 - \frac{k_1\gamma_1}{\gamma_0} zB_0 = -(1-z). \end{aligned}$$

3. Continued fractions which converge for all z to the same func-

tion. The convergence of the continued fraction (1.1) corresponding to the series (1.2) is considered first to the function $1/(1-z)$ (Theorem 3.1) and second to a meromorphic function of $1/z$ (Theorem 3.2). For this purpose, let

$$\prod_{i=0}^{p}(1-\gamma_i\bar{\gamma}_i) = U_p, \qquad p = 0, 1, \cdots.$$

Then, by (2.2), $k_p\gamma_p/\gamma_{p-1} = -1/U_{p-1}$, $p=1, 2, \cdots$. Formulas (2.5) and (2.6) can be written in the form

$$(3.1)\qquad \begin{aligned}(-1)^pA_{2p} &= \frac{A_0z^p}{U_0U_1\cdots U_{p-1}} \\ &\quad - P\cdot\left[1+\frac{z}{U_{p-1}}+\frac{z^2}{U_{p-1}U_{p-2}}+\cdots+\frac{z^{p-1}}{U_{p-1}U_{p-2}\cdots U_1}\right]\end{aligned}$$

and

$$(3.2)\qquad \begin{aligned}\frac{(-1)^pU_0U_1\cdots U_{p-1}A_{2p}}{z^p} &= A_0 - P\cdot\left[\frac{U_0}{z}+\frac{U_0U_1}{z^2}+\cdots\right. \\ &\qquad \left.+\frac{U_0U_1\cdots U_{p-1}}{z^p}\right], \qquad p = 1, 2, \cdots,\end{aligned}$$

respectively.

THEOREM 3.1. *If* $|U_p| \geqq g(p+1)$, $p=0, 1, \cdots$, *where* g *is a positive constant, then, for the continued fraction* (1.1) *corresponding to the geometric series* (1.2), *the even part converges to* $1/(1-z)$. *If, furthermore,* $|U_p| \geqq [(p+1)/p]^{1+r}|U_{p-1}|$ *for* p *sufficiently large, where* r *is an arbitrarily small positive constant, and consequently* $|U_p| > g(p+1)^{1+r}$, *then the odd part also converges to* $1/(1-z)$, $z\neq 0$. (*For* $z=1$, *the continued fraction converges in the wider sense* (*cf.* [2, *p.* 232]).

PROOF. From (3.1), $\lim_{p\to\infty}(-1)^pA_{2p} = 0 - P(1+0) = 1$. Similarly, $\lim_{p\to\infty}(-1)^pB_{2p} = 0 - Q(1+0) = 1-z$, so that $\lim_{p\to\infty} A_{2p}/B_{2p} = 1/(1-z)$. Furthermore,

$$\begin{aligned}(-1)^{p-1}A_{2p-2} &= \frac{A_0z^{p-1}}{U_0U_1\cdots U_{p-2}} \\ &\quad - P\cdot\left[1+\frac{z}{U_{p-2}}+\frac{z^2}{U_{p-2}U_{p-3}}+\cdots+\frac{z^{p-2}}{U_{p-2}U_{p-3}\cdots U_1}\right],\end{aligned}$$

and with the use of (2.7)

$$(-1)^p k_p \gamma_p A_{2p-1} = (-1)^p [A_{2p} + A_{2p-2}] = \frac{A_0 z^{p-1}}{U_0 U_1 \cdots U_{p-1}} [z - U_{p-1}]$$
$$- P \cdot \left[z \cdot \frac{U_{p-2} - U_{p-1}}{U_{p-1} U_{p-2}} + z^2 \cdot \frac{U_{p-3} - U_{p-1}}{U_{p-1} U_{p-2} U_{p-3}} \right.$$
$$+ \cdots + z^{p-2} \cdot \frac{U_1 - U_{p-1}}{U_{p-1} U_{p-2} \cdots U_1}$$
$$\left. + z^{p-1} \cdot \frac{1}{U_{p-1} U_{p-2} \cdots U_1} \right].$$

When one divides by $(U_{p-2} - U_{p-1})/U_{p-1}U_{p-2}$ and takes the limit,[4] then $\lim_{p\to\infty} (-1)^p k_p \gamma_p A_{2p-1} \cdot U_{p-1}U_{p-2}/(U_{p-2} - U_{p-1}) = 0 - P(z+0) = z$. Similarly, $\lim_{p\to\infty} (-1)^p k_p \gamma_p B_{2p-1} \cdot U_{p-1}U_{p-2}/(U_{p-2} - U_{p-1}) = 0 - Q(z+0) = (1-z)z$. Therefore, $\lim_{p\to\infty} A_{2p-1}/B_{2p-1} = 1/(1-z)$.

As an illustration of this theorem, let $\gamma_p = \gamma$, $1 - \gamma\bar{\gamma} = t$, $|t| > 1$; then $k_0 = 1/\gamma$, $k_p = -1/t^p$, $p = 1, 2, \cdots$. Here the continued fraction converges to $1/(1-z)$.

THEOREM 3.2. *If* $|U_p| \leqq G/(p+1)$, $p = 0, 1, \cdots$, *where* G *is a positive constant, then the continued fraction* (1.1) *corresponding to the geometric series* (1.2) *converges to a meromorphic function of* $1/z$ (*with the origin as an essential singular point*).

PROOF. $\lim_{p\to\infty} (-1)^p U_0 U_1 \cdots U_{p-1} A_{2p}/z^p = A_0 - P \cdot E(1/z)$, from (3.2), where $E(1/z) = U_0/z + U_0U_1/z^2 + \cdots + U_0U_1 \cdots U_{p-1}/z^p + \cdots$ is an entire transcendental function of $1/z$ for which $|G/z| + (1/2!)|G/z|^2 + (1/3!)|G/z|^3 + \cdots$ is a majorant series. An analogous formula holds for B_{2p}, and consequently

$$\lim_{p\to\infty} \frac{A_{2p}}{B_{2p}} = \frac{A_0 - P \cdot E(1/z)}{B_0 - Q \cdot E(1/z)} = \frac{1 + E(1/z)}{1 + (1-z) \cdot E(1/z)}.$$

Furthermore,

$$\lim_{p\to\infty} \frac{(-1)^p U_0 U_1 \cdots U_{p-1} A_{2p-2}}{z^p}$$
$$= \lim_{p\to\infty} \frac{(-1)^{p-1} U_0 U_1 \cdots U_{p-2} A_{2p-2}}{z^{p-1}} \cdot \left[-\frac{U_{p-1}}{z} \right]$$
$$= \left[1 + E\left(\frac{1}{z}\right) \right] \cdot 0 = 0.$$

[4] Here the hypothesis $|U_p| \geqq [(p+1)/p]^{1+r} |U_{p-1}|$ is necessary so that one can prove the series has the limit $z+0$.

Therefore,

$$\lim_{p\to\infty} \frac{(-1)^p U_0 U_1 \cdots U_{p-1} k_p \gamma_p A_{2p-1}}{z^p}$$

$$= \lim_{p\to\infty} \frac{(-1)^p U_0 U_1 \cdots U_{p-1} A_{2p}}{z^p} = 1 + E\left[\frac{1}{z}\right].$$

Similarly,

$$\lim_{p\to\infty} \frac{(-1)^p U_0 U_1 \cdots U_{p-1} k_p \gamma_p B_{2p-1}}{z^p} = 1 + (1-z)\cdot E\left[\frac{1}{z}\right],$$

and hence

$$\lim_{p\to\infty} \frac{A_{2p-1}}{B_{2p-1}} = \frac{1 + E(1/z)}{1 + (1-z)\cdot E(1/z)}.$$

Consequently, the continued fraction converges to the meromorphic function of $1/z$, $(1+E(1/z))/(1+(1-z)\cdot E(1/z))$.

When $|1-\gamma_p\bar{\gamma}_p| \leqq m < 1$, one obtains a special case of the above theorem. For example, let $\gamma_p = \gamma$, $1-\gamma\bar{\gamma} = t$, $|t| < 1$; then $k_0 = 1/\gamma$, $k_p = -1/t^p$, $p = 1, 2, \cdots$. By (3.2) for all z,

$$\lim_{p\to\infty} \frac{A_{2p}}{B_{2p}} = \lim_{p\to\infty} \frac{A_{2p-1}}{B_{2p-1}}$$

$$= \frac{1 + \dfrac{t}{z} + \dfrac{t^3}{z^2} + \dfrac{t^6}{z^3} + \cdots + \dfrac{t^{i(i+1)/2}}{z^i} + \cdots}{1 + (1-z)\left[\dfrac{t}{z} + \dfrac{t^3}{z^2} + \dfrac{t^6}{z^3} + \cdots + \dfrac{t^{i(i+1)/2}}{z^i} + \cdots\right]}.$$

4. Continued fractions which converge to different functions in different regions. If all k_p are equal ($k_p = k$), then it is advantageous if one writes k in the form $k = 2\cos\alpha\cdot e^{i\phi}$, where $0 \leqq \alpha < \pi/2$ or α is pure imaginary. One finds then from (2.3) or from the recurrence formula

$$\frac{\bar{\gamma}_0}{\gamma_{p-1}} + \frac{1}{\gamma_p} = -\frac{\gamma_0}{\gamma_{p+1}}, \qquad p = 0, 1, \cdots,$$

which can be proved by induction with the help of (2.3), that

$$\gamma_p = (-1)^p \frac{\sin\alpha}{\sin(p+2)\alpha} e^{-(p+1)i\phi}, \quad p = 0, 1, \cdots.$$

It follows that α/π if real must either be 0 or irrational, in order that

the continued fraction (1.1) actually exists. If one substitutes this value of γ_p in (2.5) and (2.6), one finds that the continued fraction (1.1) corresponding to the series (1.2) converges to $1/(1-z)$ for small values of $|z|$ and to the function $|k|^2-1/z$ for large values of $|z|$. For $|k|>2$ there exists an annulus where the continued fraction diverges. The computation is easiest for $k=2$ and this case will be considered in Theorem 4.1. The general case $k_p=k$ will be treated in Theorems 4.2 and 4.3.

THEOREM 4.1. *For $0<|z|<1/2$ and for $z=1/2$, the continued fraction* (1.3) (*where $k_p=2$, $\gamma_p=(-1)^p/(p+2)$, $p=0, 1, \cdots$, in* (1.1))[5] *converges to the value $1/(1-z)$. For $|z|>1/2$,* (1.3) *converges to the value $4-1/z$. For $|z|=1/2$, $z\neq 1/2$, the continued fraction diverges in such a manner that the even part converges to $1/(1-z)$ and the odd part to $4-1/z$.*

PROOF. From (2.5) and (2.8),

$$(-1)^p A_{2p} = \frac{2A_0}{p+2}(2z)^p - P\cdot\left[1+\frac{p+1}{p+2}(2z)+\frac{p}{p+2}(2z)^2 + \cdots + \frac{3}{p+2}(2z)^{p-1}\right], \tag{4.1}$$

or, by summation of the terms in the bracket,

$$(-1)^p A_{2p} = \frac{2A_0}{p+2}(2z)^p - \frac{P}{(1-2z)^2}\cdot\left[1-\frac{p+3}{p+2}(2z) - \frac{2}{p+2}(2z)^p + \frac{3}{p+2}(2z)^{p+1}\right]. \tag{4.2}$$

From (4.2) for $|2z|\leqq 1$, but $2z\neq 1$,

$$\lim_{p\to\infty}(-1)^p A_{2p} = (1-2z)/(1-2z)^2.$$

Similarly,

$$\lim_{p\to\infty}(-1)^p B_{2p} = (1-z)(1-2z)/(1-2z)^2,$$

and therefore

$$\lim_{p\to\infty} A_{2p}/B_{2p} = 1/(1-z).$$

[5] In the more general case $|k|=2$, let $k=2e^{i\phi}$. Then $\gamma_p=((-1)^p/(p+2))e^{-(p+1)i\phi}$, and the resulting continued fraction is equivalent to the one for which $k=2$.

From (4.2) for $|2z|>1$,

$$\lim_{p\to\infty}\frac{(-1)^p A_{2p}\cdot p}{(2z)^p} = 2A_0 - P\cdot\frac{-2+3\cdot 2z}{(1-2z)^2} = \frac{8z^2-2z}{(1-2z)^2}\cdot$$

Similarly,

$$\lim_{p\to\infty}\frac{(-1)^p B_{2p}\cdot p}{(2z)^p} = 2B_0 - Q\cdot\frac{-2+3\cdot 2z}{(1-2z)^2} = \frac{2z^2}{(1-2z)^2};$$

consequently,

$$\lim_{p\to\infty}\frac{A_{2p}}{B_{2p}} = \frac{8z^2-2z}{2z^2} = 4-\frac{1}{z}\cdot$$

To compute the odd part of (1.3), one forms first with the use of (2.7) and (4.2) the relation

$$\begin{aligned}(-1)^p 2\gamma_p A_{2p-1} &= 2A_0(2z)^{p-1}\left[\frac{2z}{p+2}-\frac{1}{p+1}\right]\\ &-\frac{P}{(1-2z)^2}\cdot\left[\frac{2z}{(p+1)(p+2)}+\frac{(2z)^{p-1}}{p+1}(2-3\cdot 2z)\right.\\ &\left.-\frac{(2z)^p}{p+2}(2-3\cdot 2z)\right].\end{aligned}\tag{4.3}$$

Then, if $|2z|<1$, from (4.3),

$$\lim_{p\to\infty}(-1)^p 2\gamma_p A_{2p-1}p^2 = \frac{-P}{(1-2z)^2}\cdot 2z = \frac{2z}{(1-2z)^2}\cdot$$

Similarly,

$$\lim_{p\to\infty}(-1)^p 2\gamma_p B_{2p-1}p^2 = \frac{-Q}{(1-2z)^2}\cdot 2z = \frac{(1-z)\cdot 2z}{(1-2z)^2},$$

$$\lim_{p\to\infty}\frac{A_{2p-1}}{B_{2p-1}} = \frac{1}{1-z}\cdot$$

If $|2z|\geqq 1$ but $2z\neq 1$, from (4.3),

$$\begin{aligned}&\lim_{p\to\infty}(-1)^p 2\gamma_p A_{2p-1}\cdot\frac{p}{(2z)^{p-1}}\\ &\quad = 2A_0(2z-1) - \frac{P}{(1-2z)^2}(1-2z)(2-6z) = \frac{2z-8z^2}{1-2z}\cdot\end{aligned}$$

Also

$$\lim_{p\to\infty} (-1)^p 2\gamma_p B_{2p-1}\cdot\frac{p}{(2z)^{p-1}}$$
$$= 2B_0(2z-1) - \frac{Q}{(1-2z)^2}(1-2z)(2-6z) = \frac{-2z^2}{1-2z},$$

and

$$\lim_{p\to\infty}\frac{A_{2p-1}}{B_{2p-1}} = \frac{2z-8z^2}{-2z^2} = 4 - \frac{1}{z}\cdot$$

If $2z=1$, from (4.1),

$$(-1)^p A_{2p} = \frac{2}{p+2}A_0 - P\cdot\left[\frac{p+3}{2} - \frac{3}{p+2}\right]. \tag{4.4}$$

Consequently,

$$\lim_{p\to\infty}\frac{(-1)^p A_{2p}}{p} = -P\cdot\frac{1}{2}, \qquad \lim_{p\to\infty}\frac{(-1)^p B_{2p}}{p} = -Q\cdot\frac{1}{2},$$

and

$$\lim_{p\to\infty}\frac{A_{2p}}{B_{2p}} = \frac{1/2}{(1-z)/2} = \frac{1}{1-z} = 2.$$

Finally, for $2z=1$, from (2.7) and (4.4),

$$(-1)^p 2\gamma_p A_{2p-1} = 2A_0\left[\frac{1}{p+2} - \frac{1}{p+1}\right]$$
$$- P\cdot\left[\frac{p+3}{2} - \frac{3}{p+2} - \frac{p+2}{2} + \frac{3}{p+1}\right].$$

Then,

$$\lim_{p\to\infty}(-1)^p 2\gamma_p A_{2p-1} = 0 - P\cdot\frac{1}{2} = \frac{1}{2},$$

$$\lim_{p\to\infty}(-1)^p 2\gamma_p B_{2p-1} = (1-z)\cdot\frac{1}{2} = \frac{1}{4},$$

$$\lim_{p\to\infty}\frac{A_{2p-1}}{B_{2p-1}} = 2.$$

This completes the proof of Theorem 4.1.

THEOREM 4.2. *If* $k_p = k$, $|k| < 2$, *the continued fraction* (1.1) *corresponding to the series* (1.2) *converges to* $1/(1-z)$ *for* $|kz| < 1$, z *not real, and to* $|k|^2 - 1/z$ *for* $|kz| > 1$, z *not real.*

THEOREM 4.3. *If* $k_p = k = 2 \cosh \beta$ $(\beta > 0)$, *the continued fraction* (1.1) *corresponding to the series* (1.2) *converges to* $1/(1-z)$ *for* $|kz| < e^{-\beta}$ *and to* $|k|^2 - 1/z$ *for* $|kz| > e^{\beta}$. *In the annulus* $e^{-\beta} < |kz| < e^{\beta}$, *the continued fraction diverges in such a manner that the even part converges to* $1/(1-z)$ *and the odd part to* $|k|^2 - 1/z$.

PROOF OF THEOREMS 4.2 AND 4.3. If

$$k_p = k = 2 \cos \alpha \cdot e^{i\phi},$$

$$\gamma_p = (-1)^p \frac{\sin \alpha}{\sin (p+2)\alpha} \cdot e^{-(p+1)i\phi}, \qquad p = 0, 1, \cdots, \tag{4.5}$$

then (2.5), after multiplication by $(-1)^p \sin (p+2)\alpha$ and with the use of the abbreviation

$$2z \cos \alpha = |k| z = x, \tag{4.6}$$

can be written in the form

$$(-1)^p A_{2p} \sin (p+2)\alpha = \sin 2\alpha \cdot x^p A_0 - P \cdot [\sin (p+2)\alpha + \sin (p+1)\alpha \cdot x + \sin p\alpha \cdot x^2 + \cdots + \sin 3\alpha \cdot x^{p-1}].$$

After summation of the terms in the bracket, this becomes

$$(-1)^p A_{2p} \sin (p+2)\alpha = \sin 2\alpha \cdot x^p A_0 - P \cdot \left[\frac{\sin (p+2)\alpha - x \sin (p+3)\alpha - x^p \sin 2\alpha + x^{p+1} \sin 3\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})} \right].$$

This can be written in the form

$$(-1)^p A_{2p} \sin (p+2)\alpha = x^p \left[A_0 \sin 2\alpha + P \cdot \frac{\sin 2\alpha - x \sin 3\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})} \right] - P \cdot \frac{\sin (p+2)\alpha - x \sin (p+3)\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})}$$

or

$$(-1)^p A_{2p} \sin (p+2)\alpha = x^p \cdot K - P \cdot \frac{\sin (p+2)\alpha - x \sin (p+3)\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})}, \tag{4.7}$$

where

$$(4.8)\quad K = A_0 \sin 2\alpha + P\cdot\frac{\sin 2\alpha - x \sin 3\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})} = \frac{\sin 2\alpha\cdot(x^2 - z)}{(x - e^{i\alpha})(x - e^{-i\alpha})}.$$

Similarly, the notation

$$(4.9)\quad L = B_0 \sin 2\alpha + Q\cdot\frac{\sin 2\alpha - x \sin 3\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})} = \frac{\sin 2\alpha\cdot z^2}{(x - e^{i\alpha})(x - e^{-i\alpha})}$$

will be used in connection with the B_p. From formula (4.7), one obtains

$$(4.10)\quad \begin{aligned}&(-1)^{p-1}A_{2p-2} \sin (p + 1)\alpha \\ &\qquad = x^{p-1}\cdot K - P\cdot\frac{\sin (p + 1)\alpha - x \sin (p + 2)\alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})}.\end{aligned}$$

When one multiplies (4.7) by $\sin (p+1)\alpha$, (4.10) by $\sin (p+2)\alpha$, by subtraction and the use of the formula $\sin (p+3)\alpha\cdot\sin (p+1)\alpha - \sin^2 (p+2)\alpha = -\sin^2 \alpha$, one finds that

$$(4.11)\quad \begin{aligned}&(-1)^p k\gamma_p A_{2p-1} \sin (p + 1)\alpha\cdot\sin (p + 2)\alpha \\ &\qquad = x^{p-1}\cdot K[x \sin (p + 1)\alpha - \sin (p + 2)\alpha] \\ &\qquad\quad - P\cdot\frac{x \sin^2 \alpha}{(x - e^{i\alpha})(x - e^{-i\alpha})}.\end{aligned}$$

One obtains analogous expressions for B_{2p} and B_{2p-1} if one replaces P with Q and K with L in (4.7) and (4.11), respectively.

The proof of Theorem 4.2 will now be completed. For α real, i.e., $|k| < 2$, if $|x| = |kz| < 1$ and x is not real, then the absolute value of the coefficient of P in (4.7) exceeds a positive bound for all values of p. Therefore, after dividing by this coefficient, one obtains as $p \to \infty$ the limit $-P$ for the right-hand member of (4.7). For real values of x, the numerator of the coefficient of P in (4.7) for certain values of p becomes arbitrarily small in absolute value, so that after dividing by this coefficient one cannot be certain about the value of the limit. Similarly, the analogous formula for B_{2p} has the limit $-Q$, and by simpler considerations without the exclusion of the real values of x one obtains the limits for A_{2p-1} and B_{2p-1} from (4.11). Hence

$$\lim_{p\to\infty} \frac{A_{2p}}{B_{2p}} = \lim_{p\to\infty} \frac{A_{2p-1}}{B_{2p-1}} = \frac{P}{Q} = \frac{1}{1 - z}.$$

If $|x| = |kz| > 1$, and x is not real, from (4.11), after dividing by $x^{p-1}[x \sin (p+1)\alpha - \sin (p+2)\alpha]$, one obtains the limit K. For real

values of x, the quantity $x \sin (p+1)\alpha - \sin (p+2)\alpha$ becomes arbitrarily small in absolute value for certain values of p, so that after division by this coefficient it is possible that one may not obtain the limit K. For the even part the considerations in (4.7) are simpler and the exclusion of the real values of x is not necessary. Therefore,

$$\lim_{p\to\infty} \frac{A_{2p}}{B_{2p}} = \lim_{p\to\infty} \frac{A_{2p-1}}{B_{2p-1}} = \frac{K}{L} = 4\cos^2\alpha - \frac{1}{z} = |k|^2 - \frac{1}{z}.$$

To complete the proof of Theorem 4.3, one writes $\alpha = i\beta (\beta > 0)$ in order that $|k| > 2$. Then (4.7) and (4.11) can be written in the form

$$\begin{aligned}(-1)^p A_{2p} \sinh (p+2)\beta \\ = x^p \cdot K' - P \cdot \frac{\sinh (p+2)\beta - x \sinh (p+3)\beta}{(x - e^{\beta})(x - e^{-\beta})}\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}(-1)^p k\gamma_p A_{2p-1} \sinh (p+1)\beta \sinh (p+2)\beta \\ = x^{p-1} \cdot K' [x \sinh (p+1)\beta - \sinh (p+2)\beta] \\ - P \cdot \frac{x \sinh^2 \beta}{(x - e^{\beta})(x - e^{-\beta})},\end{aligned} \tag{4.13}$$

respectively, where

$$K' = \frac{\sinh 2\beta \cdot (x^2 - z)}{(x - e^{\beta})(x - e^{-\beta})}, \qquad L' = \frac{\sinh 2\beta \cdot z^2}{(x - e^{\beta})(x - e^{-\beta})}.$$

If $|x| = |kz| < e^{\beta}$, $x \neq e^{-\beta}$, after dividing (4.12) by $e^{\beta p}/2$, one obtains

$$\lim_{p\to\infty} \frac{(-1)^p A_{2p} \sinh (p+2)\beta}{e^{\beta p}/2} = 0 - P \cdot \frac{e^{2\beta} - xe^{3\beta}}{(x - e^{\beta})(x - e^{-\beta})} = \frac{Pe^{3\beta}}{x - e^{\beta}}.$$

Similarly,

$$\lim_{p\to\infty} \frac{(-1)^p B_{2p} \sinh (p+2)\beta}{e^{\beta p}/2} = \frac{Qe^{3\beta}}{x - e^{\beta}}.$$

Consequently,

$$\lim_{p\to\infty} \frac{A_{2p}}{B_{2p}} = \frac{P}{Q} = \frac{1}{1 - z}.$$

If $|x| = |kz| > e^{\beta}$, when one divides (4.12) by x^p, the coefficient of P approaches zero as $p \to \infty$ and the limit of the right-hand member is

K'. The limit of the analogous expression for B_{2p} is L'. Therefore, $\lim_{p\to\infty} A_{2p}/B_{2p} = K'/L' = |k|^2 - 1/z$.

To compute the odd part, one obtains from (4.13), for $|x| = |kz| < e^{-\beta}$, $\lim_{p\to\infty} (-1)^p k\gamma_p A_{2p-1} \sinh (p+1)\beta \sinh (p+2)\beta = 0 - P\cdot x \sinh^2 \beta/(x-e^{\beta})(x-e^{-\beta})$. A similar expression holds for B_{2p-1} with P replaced by Q. Hence $\lim_{p\to\infty} A_{2p-1}/B_{2p-1} = P/Q = 1/(1-z)$. If $|x| = |kz| > e^{-\beta}$, $x \neq e^{\beta}$, one obtains from (4.13)

$$\lim_{p\to\infty} \frac{(-1)^p k\gamma_p A_{2p-1} \sinh (p+1)\beta \sinh (p+2)\beta}{x^{p-1}e^{\beta p}/2} = K'[xe^{\beta} - e^{2\beta}] + 0.$$

The corresponding expression involving B_{2p-1} holds with K' replaced by L'. Consequently,

$$\lim_{p\to\infty} \frac{A_{2p-1}}{B_{2p-1}} = \frac{K'}{L'} = \frac{x^2 - z}{z^2} = |k|^2 - \frac{1}{z}.$$

This completes the proof of Theorem 4.3.

REMARK. As already noticed in the proof, in Theorem 4.2 for $|kz| < 1$ the odd part also converges for real values of z to $1/(1-z)$. Likewise for $|kz| > 1$ the even part also converges for real values of z to $|k|^2 - 1/z$. The question whether for $|kz| < 1$ the even part and for $|kz| > 1$ the odd part also converge for real values of z would be a difficult problem in Diophantine approximations.

5. **Observation concerning the values to which the continued fraction (1.1) converges.** The above properties of the continued fraction (1.1) are connected with the following fact not already remarked in the discussion.

The approximants of (1.1) can be expanded in power series not only in *ascending* powers of z but also in *descending* powers of z. As one can verify, corresponding to the continued fraction (1.1) there exists a power series in descending powers of z, $d_0 + d_1/z + d_2/z^2 + \cdots$, which has the property that the approximants A_{2p-1}/B_{2p-1} and A_{2p}/B_{2p} agree with the series up to and including the term involving $1/z^{p-1}$, $p = 1, 2, \cdots$. One would consider it normal if the continued fraction would be equal for small values of $|z|$ to the power series in ascending powers of z and equal for large values of $|z|$ to the series in descending powers of z. This is, for example, the case in Theorems 4.1, 4.2, and 4.3. On the contrary, in Theorem 3.1 the continued fraction is everywhere equal to the series in ascending powers of z and to its analytic continuation, and in Theorem 3.2 the continued fraction is everywhere equal to the series in descending powers of z and to its analytic continuation.

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A REPRESENTATION PROBLEM[1]

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1. **Introduction and summary.** The problem solved in this paper is the following. For a fixed number a, $0<a<1$, what functions $f(x)$ on $0\leqq x\leqq 1$ have a representation

$$f(x) = \sum_1^\infty c_n\phi_n(x), \tag{1}$$

where $c_n\geqq 0$, $\sum_1^\infty c_n$ converges, and each ϕ_n is the characteristic function of a subset of $0\leqq x\leqq 1$ of Lebesgue measure a? Clearly any f satisfying (1) satisfies

$$0 \leqq f(x) \leqq \frac{1}{a}\int_0^1 f(x)dx \qquad \text{for all } x, \tag{2}$$

since $f(x)\leqq \sum_1^\infty c_n=(1/a)\int_0^1 f(x)dx$. The result of this paper is that (2) *is sufficient as well as necessary for a function f to admit a representation* (1).

The problem arises in connection with a submarine search game discussed by Morse and Kimball [1], which runs as follows. A submarine must pass through a channel of length 1, and must choose a part of the channel of length a at which to be on the surface. An enemy plane must choose a point x along the channel at which to look for the submarine. The submarine escapes if it is submerged at the x chosen by the plane; otherwise it has a probability $h(x)$ of being detected. Thus a pure strategy for the submarine is a subset S of $0\leqq x\leqq 1$ of Lebesgue measure a, and its detection probability for the given S as a function of x is $h(x)\phi(x)$, where ϕ is the characteristic

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