NOTE ON A CONVERSE OF LUCAS'S THEOREM

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1. Let $p(z)$ be a polynomial of positive degree. The convex closure of the roots of $p(z)$ will be called the *Lucas polygon* π of $p(z)$. If $p'(z)$ is the derivative of $p(z)$ and π' is its Lucas polygon, a theorem of Lucas implies that $\pi' \subset \pi$ [1, p. 6]. Suppose we are given $p'(z)$ and its corresponding π'. We wish to determine connections between π' and π_0, the intersection of all the Lucas polygons of the primitives $p(z)+c$ of $p'(z)$. It is obvious that π_0 is compact, convex, and contains π'. However π_0 need not coincide with π' [1, p. 72]. Each point of the boundary of π_0 is a boundary point of the Lucas polygon $\pi(c)$ corresponding to some $p(z)+c$ (see below). We find some necessary conditions on $p(z)+c$ that its $\pi(c)$ have a boundary point in common with π_0.

2. A side of a Lucas polygon $\pi(c)$ will be called *simple* if it is determined by just two simple roots of $p(z)+c$. We have the following

THEOREM. *A simple side of $\pi(c)$ has no point in common with the boundary of π_0 unless the degree of $p'(z)=1$.*

PROOF. Since π_0 is a compact, convex set in the plane, its set of boundary points forms a simple closed curve B. If $q_0 \in B$ there exists a sequence of points $\{q_n\}$ in the complement of π_0 with $\lim_{n\to\infty} q_n = q_0$ and a sequence $\{c_n\}$ such that if $\pi(c_n)$ is the Lucas polygon of $p(z)+c_n$, $q_n \notin \pi(c_n)$. It is clear from Theorem 1, p. 21 of [1] that the sequence $\{c_n\}$ is bounded and therefore has a limit point c_0. Since the roots of a polynomial are continuous functions of the coefficients, q_0 is on the boundary of the Lucas polygon $\pi(c_0)$ of $p(z)+c_0$.

Suppose q_0 is on a simple side L of $\pi(c_0)$ and let ζ_0 and η_0 be the simple roots of $p(z)+c_0$ which determine L. Since ζ_0 and η_0 are the only roots on the side L, a sufficiently small change in c_0 to a value c will vary ζ_0 and η_0 to roots ζ and η which again determine a simple side. Also we have $p'(\zeta_0)\neq 0$, $p'(\eta_0)\neq 0$. As the coefficient c changes, ζ and η change and are analytic functions of c given by equations of the form

$$\zeta = \zeta_0 + (1/p'(\zeta_0))(c - c_0) + \cdots ,$$
$$\eta = \eta_0 + (1/p'(\eta_0))(c - c_0) + \cdots .$$

We may write $q_0 = a\zeta_0 + b\eta$ where a and b are non-negative real num-

Received by the editors March 3, 1953 and, in revised form, July 9, 1953.

bers whose sum is one. Putting $q=a\zeta+b\eta$, we see that q is an analytic function of c. If this function q were constant, $aD_c^{(n)}\zeta+bD_c^{(n)}\eta=0$ for every $n>0$, or assuming $a\neq 0$, $D_c^{(n)}\zeta=kD_c^{(n)}\eta$ $(k=-b/a)$. This would mean that our two series are of the form

$$\zeta = \zeta_0 + k\sum_1^\infty a_n(c-c_0)^n, \qquad \eta = \eta_0 + \sum_1^\infty a_n(c-c_0)^n$$

where $a_1\neq 0$. Now $p(\zeta)+c\equiv p(\eta)+c\equiv 0$ from which $p(\zeta_0+kw)\equiv p(\eta_0+w)$ where w is free to vary in a neighborhood of zero. But $p(z)$ is a polynomial of degree n so this last identity requires that $k^n=1$. Since k is also real, $k=1$ if n is odd and $k=1$ or -1 if n is even. For $k=1$ we get $p(\zeta_0+w)=p(\eta_0+w)$ which means that $p(z)$ must be a periodic function. Since $p(z)$ is also a polynomial we have a contradiction. Thus in this case q is a nonconstant function of c and the mapping from the c-plane to the q-plane is open. Hence q_0 cannot be a boundary point of π_0 in this case.

Finally if $k=-1$ we have $p(\zeta_0+w)=p(\eta_0-w)$ or equivalently

$$p((\zeta_0+\eta_0)/2+(w-(\eta_0-\zeta_0)/2)) = p((\zeta_0+\eta_0)/2-(w-(\eta_0-\zeta_0)/2)).$$

From this last identity it follows that the odd derivatives of $p(z)$ vanish for $z=(\zeta_0+\eta_0)/2$. Since ζ_0 and η_0 determine a simple side of $p(z)+c$, degree $p'(z)=1$ completing the proof of the theorem (see the statement of Lucas's theorem on p. 6 of [1]).

3. The sides of a Lucas polygon π which are not simple either have a multiple root on them or have more than two simple roots. We give some examples to show some of the problems involved in finding more precise information on π_0. If $p(z)$ is a polynomial with multiple roots at all the vertices of its Lucas polygon, then $p'(z)$ has the same Lucas polygon as $p(z)$. In this case π_0 coincides with π'. That this is not true in general is shown on p. 72 of [1]. If $p'(z)$ is of degree one or two or of degree three with collinear roots, then $\pi'=\pi_0$ [1, p. 71].

If $p'(z)$ is of degree three and its roots are not collinear then $\pi'\neq\pi_0$ in general (see [1, p. 72]). The vertices of π' are on the boundary of π_0 however. This is not true in general as the following example shows. Consider the polynomial $p_1(z)=z^2(z+1)(z^2+1)$. $p_1'(z)$ has four roots, one at zero, one a negative real number, and two with negative real parts. Let $p(z)=z^2(z+1)(z-2\alpha z+1+\alpha^2)$ where α is a positive real number small enough so that the complex roots of $p'(z)$ still have negative real parts. The Lucas polygon π' of $p'(z)$ has zero as a vertex. However zero is not on the boundary of π_0. For the

Lucas polygon of the primitive having zero as a multiple root has as a side the segment $(-i+\alpha, i+\alpha)$ and zero is in its interior. It is clear that this is the only primitive that could have zero on the boundary of its Lucas polygon and since it fails to have it there zero must be in the interior of π_0.

It is evident from our theorem that the primitives with multiple roots play an important role in the determination of π_0. Since two primitives with a root in common are identical, there are just a finite number of primitives with multiple roots (less than or equal to the degree of $p'(z)$). However even if $p'(z)$ is a cubic, π_0 need not be completely determined by the primitives with multiple roots as the following example shows. Let $p'(z)=4z^3+(9/2)z^2+2z+3/2$. The roots of $p'(z)$ are -1 and $(-1\pm(95)^{1/2}i)/16$ and these are the vertices of π'. The three primitives of $p'(z)$ with multiple roots are

$$z^4 + (3/2)z^3 + z^2 + (3/2)z + 1 = (z+1)^2(z-(1/2)z+1),$$

$$\begin{aligned} z^4 + (3/2)z^3 + z^2 + (3/2)z &+ (1867 - 475(95)^{1/2}i)/8192 \\ &= [z + (1-(95)^{1/2}i)/16]^2[z^2 + ((11+(95)^{1/2}i)/8)z \\ &\quad + (-37 + 21(95)^{1/2}i)/128], \end{aligned}$$

$$\begin{aligned} z^4 + (3/2)z^3 + z^2 + (3/2)z &+ (1867 + 475(95)^{1/2}i)/8192 \\ &= [z + (1+(95)^{1/2}i)/16]^2[z^2 + ((11-(95)^{1/2}i)/8)z \\ &\quad + (-37 - 21(95)^{1/2}i)/128]. \end{aligned}$$

It can be readily verified that the intersection of the Lucas polygons of these polynomials contains zero in its interior. The primitive $z^4+(3/2)z^3+z^2+(3/2)z=z(z+3/2)(z^2+1)$ contains zero on the boundary of its Lucas polygon. Hence π_0 for this $p'(z)$ is smaller than the intersection of the Lucas polygons of the primitives with multiple roots.

Reference

1. J. L. Walsh, *The location of critical points of analytic and harmonic functions*, Amer. Math. Soc. Colloquium Publications, vol. 34, 1950.

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