ON PARACOMPACT SPACES

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1. Stone\(^2\) has proved that a space is fully normal \(T_1\) if and only if
it is paracompact \(T_2\). If throughout his proof \(T_1\) is deleted and \(T_2\)
is replaced by \(T_4\) (normality), we obtain that
\textit{a space is fully normal if and only if it is paracompact \(T_4\)}. In this note we prove that \(T_2\) can
also be replaced by any one of the following:

\(T'_2\). Every two points with disjoint closures have disjoint neigh-
borhoods.

\(T'_3\). For each point \(x\) and neighborhood \(U\) of \(x\) there is a neigh-
borhood \(V\) of \(x\) whose closure is contained in \(U\).

\(LT_4\). Every point has a neighborhood whose closure is normal.

Moreover, we shall also study a natural decomposition of a space
with certain properties and prove that a space is paracompact \(T_4\) if
and only if it has a retract which is paracompact \(T_2\) and meets every
non-null closed set.

The following terminology will be used.

\textbf{Property *}. If \(x, y, z\) are any three points such that \(x \cap y \neq \emptyset\)
and \(x \cap z \neq \emptyset\), then \(y \cap z \neq \emptyset\).

\textbf{Property **}. Every point has a compact closure.

We remark that

\begin{align}
(1) \quad & T_2 \rightarrow T'_2 \quad \text{and} \quad T_3 \rightarrow T'_3, \\
(2) \quad & T_4 \rightarrow LT_4 \rightarrow T'_3 \rightarrow T'_2 \rightarrow \text{property *}.
\end{align}

2. \textbf{Theorem 1}. \textit{In a paracompact space, \(T'_2\), \(T'_3\), \(LT_4\) and \(T_4\) are
equivalent.}

\textbf{Proof}. It is immediate that \(T_4\) implies \(T'_2\), \(T'_3\), and \(LT_4\). To prove
that \(T'_2\) plus paracompactness implies \(T'_3\) and that \(T'_3\) plus para-
compactness implies \(T_4\) we can simply follow Dieudonné's proof\(^3\)
with \(T_2\) and \(T_3\) replaced by \(T'_2\) and \(T'_3\) respectively. For any para-
compact \(LT_4\) space there is a locally finite open covering, say \(\{U_\alpha\}\),
such that each \(\overline{U_\alpha}\) is normal. Since \(\{\overline{U_\alpha}\}\) is still locally finite, the un-

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proved part of Theorem 1 follows from the following more general result.

**Lemma.** A space which is covered by a locally finite system of normal closed sets is normal.

**Proof.** Let $E$, $F$ be disjoint closed sets in a topological space $X$ covered by a locally finite system of closed normal sets $A_a$. For each $\alpha$, there exist, by the normality of $A_\alpha$, open sets $U_\alpha$, $V_\alpha$ in $X$ such that

$$E \cap A_\alpha \subset U_\alpha, \quad F \cap A_\alpha \subset V_\alpha, \quad \text{and} \quad U_\alpha \cap V_\alpha \cap A_\alpha = \emptyset.$$ 

Now we define

$$P(x) = X - \bigcup \{ A_\alpha : A_\alpha \ni x; x \in X \};$$

$$Q(x) = \bigcap \{ U_\alpha : A_\alpha \ni x; x \in E \};$$

$$R(x) = \bigcap \{ V_\alpha : A_\alpha \ni x; x \in F \}.$$ 

It follows by the local finiteness of the system $\{ A_\alpha \}$ that $P(x)$, $Q(x)$, and $R(x)$ are open. Let

$$U = \bigcup \{ P(x) \cap Q(x) : x \in E \}, \quad V = \bigcup \{ P(x) \cap R(x) : x \in F \}.$$ 

Then $U$, $V$ are disjoint neighborhoods of $E$, $F$.

**Remark.** We can show by examples that in a pointwise paracom pact space, $T_\beta$, $T_\delta$, $L T_\delta$, and $T_\delta$ are not equivalent to one another.

**Corollary.** The product of a paracompact $T_4$ space and a compact $T_4$ space is paracompact $T_4$.

To prove this corollary we have only to observe that the product of a paracompact space and a compact space is paracompact and that the product of two $T_\delta$ spaces is $T_\delta$.

3. Given any space $X$ with property $\ast$ we can define an equivalence relation such that two points $x$, $y$ of $X$ are equivalent if and only if $x \cap y \neq \emptyset$. This relation yields a decomposition $D$ of $X$, that is, a system of sets, pairwise disjoint, whose union is $X$ and such that two points of $X$ are contained in a same member of $D$ if and only if they are equivalent. Denote by $\phi$ the projection of $X$ onto $D$, i.e., the function of $X$ into $D$ such that $f(x) = \phi$ if $x \in \phi$. There is a topology on $D$ such that a subset $G$ of $D$ is open if and only if $\phi^{-1}(G)$ is open. $D$ with this topology is the natural quotient space of $X$. If $X$ is $T_4$ and then

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4 This is suggested by the referee as a known but unpublished result.


6 Cf. J. Dieudonné, loc. cit.
has property *, the natural quotient space of $X$ agrees with one constructed by Čech. Throughout the rest of this note, $D$ and $\phi$ always denote the natural quotient space and the projection for space $X$ provided $X$ has property *.

**Lemma.** Let $X$ be a space with property * and property **.

(3) Whenever $p \in D$, $\phi^{-1}(p)$ contains a smallest non-null closed set $F_p$. $F_p = \bar{x}$ for $x \in F_p$.

(4) Every open set which meets $F_p$ contains $\phi^{-1}(p)$. Hence for any $x \in X$, if $\bar{x}$ is covered by a system of open sets, $\bar{x}$ is contained in one of them.

(5) If $E$, $F$ are disjoint closed sets in $X$, then $\phi(E) \cap \phi(F) = \emptyset$.

**Proof.** Fix $p \in D$ and let $x_0 \in \phi^{-1}(p)$. Clearly $\{x \cap x_0 : x \in \phi^{-1}(p)\}$ is a system of closed sets in $\bar{x}_0$ and it has, by property *, the finite intersection property. It follows by property ** that

$$F_p = \bigcap \{x : x \in \phi^{-1}(p)\} = \bigcap \{\bar{x} \cap x_0 : x \in \phi^{-1}(p)\}$$

is a non-null closed set contained in $\phi^{-1}(p)$. For any non-null closed set $F$ contained in $\phi^{-1}(p)$, we have $F \supseteq \bar{x} \supseteq F_p$ whenever $x \in F$. Hence $F_p$ is the smallest. If $x \in F_p$, then $\bar{x} \subseteq F_p \subseteq \bar{x}$. Hence (3) is proved.

By construction $F_p \subseteq \bar{x}$ for each $x \in \phi^{-1}(p)$. Hence, if $U$ is open and meets $F_p$, $U$ meets $\{x\}$, i.e., $U$ contains each $x \in \phi^{-1}(p)$, proving that $\phi^{-1}(p) \subseteq U$. If $x \in X$ and $\bar{x}$ is covered by a system $\mathfrak{U}$ of open sets, then some open set of $\mathfrak{U}$ meets $F_p$ with $p = \phi(x)$, and hence contains $\bar{x}$. This proves (4).

If $E$ and $F$ are disjoint closed sets in $X$, then for $x \in E$ and $y \in F$, $\bar{x} \cap y \subseteq E \cap F = \emptyset$ and so $\phi(x) \neq \phi(y)$. Hence $\phi(E) \cap \phi(F) = \emptyset$, proving (5).

**Lemma.** Let $X$ be a $T_\delta'$ space with property **.

(6) $\phi$ is closed.

(7) Given any open covering $\{U_\alpha\}$ of $X$, $\{D - \phi(X - U_\alpha)\}$ is an open covering of $D$.

(8) $D$ is $T_{23}$ (i.e., $T_2$ and $T_3$).

(9) $D$ is $T_{24}$ (i.e., $T_2$ and $T_4$) or $LT_{24}$ (i.e., $T_2$ and $LT_4$) according as $X$ is $T_4$ or $LT_4$.

**Proof.** Let $F$ be a closed subset of $X$. Given any point $p$ of $D - \phi(F)$ there is, by (3), a smallest non-null closed set $F_p$ contained in $\phi^{-1}(p)$. Take a point $x_p$ of $F_p$; there is, by $T_\delta'$, a neighborhood $V_p$ of $x_p$ whose closure is contained in $X - F$. Therefore, by (4) and (5), $\phi^{-1}(p) \subseteq V_p \subseteq \overline{V_p} \subseteq X - \phi^{-1}(p) \phi(F) = \phi^{-1}(D - \phi(F))$. Hence $\phi^{-1}(D - \phi(F))$

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Given any open covering \( \{ U_a \} \) of \( X \), \( \{ D - \phi(X - U_a) \} \) is a system of open sets in \( D \) by (6). For each \( p \in D \) there is, by (4), some \( \alpha \) such that \( \phi^{-1}(\alpha) \subseteq U_a \) and so \( p \in D - \phi(X - U_a) \). Hence (7) is proved.

By (6), \( D \) is a \( T_1 \) space. Therefore \( D \) is \( T_{23} \) if we can show that for a point \( p \) of \( D \) and a neighborhood \( G \) of \( p \) there is a neighborhood of \( p \) whose closure is contained in \( G \). Take a point \( x \) of \( F_p \) and let \( U \) be a neighborhood of \( x \) with \( \overline{U} \subseteq \phi^{-1}(G) \). Then \( D - \phi(X - U) \) is a neighborhood of \( p \) by (4) and (6) and its closure is contained in \( G \) by (5).

Suppose that \( X \) is \( T_4 \). Given any two disjoint closed subsets \( P \), \( Q \) of \( D \), \( \phi^{-1}(P) \) and \( \phi^{-1}(Q) \) are disjoint closed subsets of \( X \) and they have disjoint neighborhoods \( U \) and \( V \). Applying (6), we can easily see that \( D - \phi(X - U) \) and \( D - \phi(X - V) \) are disjoint neighborhoods of \( P \) and \( Q \), proving that \( D \) is \( T_4 \) and hence \( T_{24} \).

Suppose now that \( X \) is \( LT_4 \). Let \( p \in D \) and \( x \in F_p \); there is, by hypothesis, a neighborhood \( U \) of \( x \) whose closure is normal. By (4) and (6), \( G = D - \phi(X - U) \) is a neighborhood of \( p \). Then there is, by \( T_3 \), a neighborhood \( V \) of \( p \) such that \( \overline{V} \subseteq G \). Since \( \phi^{-1}(\overline{V}) \subseteq \overline{U} \) is normal, it follows by the preceding result that \( \overline{V} \) is normal, proving that \( D \) is \( LT_4 \) and hence \( LT_{24} \).

**Lemma.** Let \( X \) be a pointwise paracompact (paracompact) space.

(10) \( X \) has property **.
(11) If \( X \) is \( T_3 \), then \( D \) is pointwise paracompact (paracompact).

**Proof.** Fix a point \( x_0 \) of \( X \). Given any system \( U \) of open sets whose union contains \( x_0 \), \( U \cup \{ X - x_0 \} \) is an open covering of \( X \) and it admits a point-finite refinement \( \mathcal{B} \). Let \( \mathcal{B}' = \{ V : V \subseteq \mathcal{B}, \; V \cap x_0 \neq \emptyset \} = \{ V : V \subseteq \mathcal{B}, \; V \supseteq x_0 \} \) and for each \( V \in \mathcal{B}' \) we take a \( U_V \in U \) such that \( V \subseteq U_V \). Then \( U' = \{ U_V : V \in \mathcal{B}' \} \) is a finite subsystem of \( U \) which covers \( x_0 \), proving the compactness of \( x_0 \). Hence \( X \) has property **.

Given any open covering \( \{ G_\alpha \} \) of \( D \), \( \phi^{-1}(G_\alpha) \) is an open covering of \( X \) and it admits, by hypothesis, a point-finite (locally finite) refinement \( \{ U_\beta \} \). By (7), \( \{ D - \phi(X - U_\beta) \} \) is an open covering of \( D \) which is obviously point-finite (locally finite) and refines \( \{ G_\alpha \} \). Hence \( D \) is pointwise paracompact (paracompact).

**Theorem 2.** A space \( X \) is pointwise paracompact (paracompact) \( T_3 \) if and only if it has a pointwise paracompact (paracompact) \( T_3 \) retract \( A \) which meets every non-null closed subset of \( X \). Moreover, such a retract \( A \) is \( LT_{24} \) or \( T_{24} \) if and only if \( X \) is \( LT_4 \) or \( T_4 \). Finally, \( A \) is unique up to a homeomorphism and the related retraction is uniquely determined.
Proof. Suppose that \( X \) is pointwise paracompact (paracompact) \( T_4' \). By (2), and (10), \( X \) has property * and property **; it follows by (3) that for each \( \rho \in \mathcal{D} \) there is a smallest non-null closed set \( F_\rho \) contained in \( \phi^{-1}(\rho) \). We take, for each \( \rho \in \mathcal{D} \), a point \( x_\rho \) of \( F_\rho \) and denote by \( A \) the set of these points \( x_\rho \). Clearly \( A \) meets every non-null closed subset of \( X \) and \( \phi \) defines a 1-1 mapping \( \psi \) of \( A \) onto \( D \). For any closed subset \( F \) of \( X \), we have \( \psi(F \cap A) = \phi(F) \) which is closed by (6). Hence \( \psi \) is a homeomorphism. From this result, we obtain a retraction \( f = \psi^{-1} \phi \) of \( X \) onto \( A \). Moreover, it follows by (8) and (9) that \( A \) is \( T_{23} \) and that \( A \) is \( T_{24} \) or \( LT_4 \) according as \( X \) is \( T_4 \) or \( LT_4 \).

Conversely suppose that \( X \) has a pointwise paracompact (paracompact) \( T_{23} \) retract \( A \) which meets every non-null closed subset of \( X \). Let \( f \) be a retraction of \( X \) into \( A \). For any \( x \in X \), \( \bar{x} \cap A \) is non-null and \( T_3 \); therefore it contains exactly one point. From this result, it is easily seen that for any two points \( x, y \) of \( X \), \( \bar{x} \cap y \neq \emptyset \) if and only if \( \bar{x} \cap A = \bar{y} \cap A \). Hence property * as well as property ** holds for \( X \), and for each \( \rho \in \mathcal{D} \), \( \phi^{-1}(\rho) \cap A \) contains exactly one point contained in \( F_\rho \). From the latter result, we have, for \( x \in X \), \( \bar{f}(x) = \bar{f}^{-1}(\rho) \cap A \).

Let \( x \in X \) and let \( U \) be a neighborhood of \( x \). Then there is a neighborhood \( G \) of \( \bar{x} \cap A \) in \( A \) such that \( \bar{x} \cap A \subset G \subset \bar{G} \cap A \subset U \cap A \). Hence \( f^{-1}(G) \) is a neighborhood of \( x \) whose closure is contained in \( U \). This proves that \( X \) is \( T_4' \).

If \( A \) is \( T_4 \), then for any two disjoint closed subsets \( E, F \) of \( X \), \( E \cap A \) and \( F \cap A \) have disjoint neighborhoods \( G \) and \( H \) in \( A \) and so \( f^{-1}(G) \) and \( f^{-1}(H) \) are disjoint neighborhoods of \( E \) and \( F \). Hence \( X \) is also \( T_4 \). If \( A \) is \( LT_4 \), then for any \( x \in X \), \( f(x) \) has a neighborhood \( G \) in \( A \) whose closure in \( A \) is normal. It follows by the preceding result that \( f^{-1}(G) \) is a neighborhood of \( x \) whose closure is normal. Hence \( X \) is also \( LT_4 \).

Given any open covering \( \{ U_\alpha \} \) of \( X \), \( \{ U_\alpha \cap A \} \) is an open covering of \( A \) and it admits, by hypothesis, a point-finite (locally finite) refinement \( \{ G_\beta \} \). We can easily see that \( \{ f^{-1}(G_\beta) \} \) is a point-finite (locally finite) refinement of \( \{ U_\alpha \} \). Hence \( X \) is pointwise paracompact (paracompact).

According to the above argument, \( A \) is homeomorphic to \( D \) and hence is unique up to a homeomorphism. Moreover, for any \( x \in X \), \( \bar{x} \cap A = f(x) \). Hence the retraction \( f \) is uniquely determined.

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