COUNTABLE PARACOMPACTNESS IN LINEARLY ORDERED SPACES

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A Hausdorff space X is said to be *paracompact* [2] provided that if W is a collection of open sets covering X, there exists a collection W' of open sets covering X such that (1) each element of W' is a subset of an element of W and (2) each point of X belongs to an open set which intersects only a finite number of elements of W'; X is said to be *countably paracompact* [3] provided that if W is a *countable* collection of open sets covering X, there exists a collection W' satisfying the above conditions. It is known that not every normal Hausdorff space is paracompact [2], but the question whether every such space is countably paracompact is as yet unsolved (cf. [3]). Since every linearly ordered space[1] is a normal Hausdorff space (cf. [1, p. 39]) but not necessarily paracompact [2], it seems natural to inquire whether every linearly ordered space must be countably paracompact. The purpose of the present note is to show that this is the case.

DEFINITIONS. 1. A collection G of subsets of a space X is said to be *locally finite* provided every point of X belongs to an open set X which intersects at most a finite number of the elements of G. 2. If G and H are collections of sets, then H is said to be a *refinement* of G provided each element of H is a subset of some element of G. 3. A collection G of sets is said to be *coherent* provided G is not the sum of two collections G_1 and G_2 such that no element of G_1 intersects an element of G_2. 4. If p belongs to some element of the collection G of sets, then the *star* of p with respect to G is the sum of the elements of G which contain p.

Suppose X is a linearly ordered space and W is a countable collection of open sets covering X. For each point p of X, let M_p denote the set of all points x of X such that there is a finite, coherent collection H_x of open intervals of X such that H_x is a refinement of W and covers the set whose elements are p and x. For each point p of X, let K_p denote the collection of all open intervals k of X such that k is a subset of M_p and of some element of W. It is easily seen that for each p in X, M_p is both open and closed and if $q \in M_p$, then $M_q = M_p$.

Presented to the Society, June 20, 1953, under the title *A note on paracompactness*; received by the editors May 14, 1953 and, in revised form, July 18, 1953.

[1] By a *linearly ordered space* is mean a simply ordered set with its intrinsic topology; i.e., the topology in which "neighborhood of x" means "open interval containing x."

Hence if for each point p of X, there is a locally finite collection G_p of open sets which is a refinement of K_p and covers M_p, then there is a locally finite collection G of open sets which is a refinement of W and covers X. The existence of such a collection G_p for each p in X is a consequence of the following two theorems.

THEOREM 1. *If $p \in X$ and $R = \{x \in M_p \mid p \leqq x\}$, then either* (1) *$R$ is covered by a finite subcollection of K_p or* (2) *there is a sequence $\{x_n\}$ of points of R such that, for each n, $x_n < x_{n+1}$ and if $x \in R$, then, for some n, $x < x_n$.*

PROOF. Let G_0 denote the star of p with respect to K_p. If there is a point of R which is preceded[2] by every point of G_0, then there is a point a_1 of R which is preceded by every point of G_0 and which belongs to an element of K_p which intersects G_0. Let G_1 denote the star of a_1 with respect to K_p. If there is a point of R which is preceded by every point of G_1, then there is a point a_2 of R which is preceded by every point of G_1 and which belongs to an element of K_p which intersects G_1. It follows by induction that there exist (possibly finite) sequences $G_0, G_1, G_2, \cdots$ and $a_0, a_1, a_2, \cdots$ $(a_0 = p)$ such that no point of R is preceded by every point of $\sum G_i$ and for each n, (1) a_n is a point of R, (2) G_n is the star of a_n with respect to K_p, and (3) a_{n+1} is preceded by every point of G_n and belongs to an element of K_p which intersects G_n. Let α denote the sequence $G_0, G_1, G_2, \cdots$.

Suppose k is an element of K_p which intersects $R \cdot \sum G_i$. Let j denote the smallest integer n such that k intersects G_n. Suppose G_j is the last term of α. Then G_j contains $\{x \in M_p \mid a_j \leqq x\}$. If $j = 0$, then, since k intersects R, k either contains a_0 or is a subset of R; in either case k is a subset of G_0. Suppose $k \neq 0$. If k contains a point x such that $a_j \leqq x$, then k is a subset of G_j. If each point of k precedes a_j, then, since there is an element of K_p which contains a_j and intersects G_{j-1} and k does not intersect G_{j-1}, k is a subset of G_j. Suppose G_j is not the last term of α. Then if k is not a subset of $G_j + G_{j+1}$, it contains a point which is preceded by every point of G_{j+1} and hence, since k intersects G_j, k contains a_{j+1}. But this implies that k is a subset of G_{j+1}. Thus every element of K_p which intersects $R \cdot \sum G_i$ is a subset of $\sum G_i$.

If α is infinite, for each n, let $x_n = a_n$; it is easily seen that $\{x_n\}$ satisfies condition (2) of the conclusion of this theorem. Suppose α is finite. Let G_j be the last term of α and let $D = \{x \in M_p \mid a_j < x\}$. Since D is an open subset of M_p, if D is a subset of any element of W, condition (1) is satisfied. Suppose D is not a subset of any element

[2] If $x < y$, then x is said to *precede* y.

of W. Let $w_1, w_2, w_3, \cdots$ denote the elements of W which contain a_j. For each n, let T_n denote the sum of all elements of K_p which contain a_j and lie in w_n. For each n, there is a point of M_p which is preceded by every point of T_n (otherwise D would be a subset of w_n) and hence there exists a sequence $\{x_n\}$ of points of R such that x_1 is preceded by every point of T_1 and for each n, x_{n+1} is preceded by each of the points $x_1, x_2, \cdots, x_n$ and by every point of T_n. Suppose x is a point of R. If x belongs to G_j, it belongs to an element k of K_p which contains a_j. Since $k \in K_p$, for some n, k is a subset of w_n and hence of T_n. Consequently $x < x_{n+1}$. If x does not belong to G_j, then $x < a_j$ and hence $x < x_1$. Hence condition (2) is fulfilled.

THEOREM 2. *Under the hypothesis of Theorem* 1, *there is a locally finite collection U of open sets which is a refinement of K_p and covers R.*

PROOF. If condition (1) of the conclusion of Theorem 1 is satisfied, there is such a collection U. Suppose Condition (1) is not satisfied and let $\{x_n\}$ be a sequence of points of R satisfying condition (2). For each n, let H_n denote a finite, coherent collection of elements of K_p which covers the set whose elements are p and x_{n+1}. Let $H_1' = H_1$ and for each n greater than 1, let H_n' denote a finite collection of open intervals such that (1) H_n' is a refinement of H_n which covers the closed interval $x_n x_{n+1}$ and (2) no element of H_n' intersects the closed interval px_{n-1}. Let $U = \sum H_i'$. Then U is a locally finite collection of open sets which is a refinement of K_p and covers R.

It can be shown by a similar argument that if $L = \{x \in M_p \mid x \leqq p\}$, then there is a locally finite collection V of open sets which is a refinement of K_p and covers L. The collection $U+V$ is a locally finite collection of open sets which is a refinement of K_p and covers M_p. It follows that X is countably paracompact.

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