ON MAXIMIZING AN INTEGRAL WITH A SIDE CONDITION

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The problem to be discussed in this paper is that of finding an admissible function, \( \phi(x) \), which makes the integral

\[
\mathcal{J}(\phi) = \int_E F(\phi(x), x) \, dx
\]

an absolute maximum subject to a side condition of the type

\[
\mathcal{G}(\phi) = \int_E G(\phi(x), x) \, dx = c.
\]

The class, \( \mathcal{P} \), of admissible functions is to include all \( \phi(x) \) satisfying

\( A_1: u(x) \leq \phi(x) \leq v(x), \)

\( A_2: F(\phi(x), x) \) and \( G(\phi(x), x) \) summable over \( E \).

It will further be assumed that

\( H_1: E \) is a compact subset of the reals,

\( H_2: u(x) \) and \( v(x) \) continuous for \( x \in E \),

\( H_3: F(\phi, x) \) and \( G(\phi, x) \) continuous for \( x \in E \) and \( u(x) \leq \phi \leq v(x) \),

\( H_4: \inf_{\phi \in \mathcal{P}} \mathcal{G}(\phi) < c < \sup_{\phi \in \mathcal{P}} \mathcal{G}(\phi). \)

Under these conditions the problem can often be solved by forming

\[
h(\theta, x) = \Max_{u(x) \leq \mu \leq v(x)} h(\theta, x, \mu)
\]

where

\[
h(\theta, x, \mu) = \cos \theta F(\mu, x) + \sin \theta G(\mu, x).
\]

If \( \theta = \theta_c \) in the open interval \( (-\pi/2, \pi/2) \) and an admissible function \( \mu_{\theta_c}(x) \) can be found such that \( \mu_{\theta_c}(x) \) maximizes \( h(\theta_c, x, \mu) \) for each \( x \), and if

\[
\mathcal{G}(\mu_{\theta_c}) = c,
\]

this function, \( \mu_{\theta_c}(x) \), will be a solution to the problem. This follows because for all \( x \in E \) and admissible \( \phi(x) \)

\[
h(\theta_c, x, \mu_{\theta_c}(x)) \geq h(\theta_c, x, \phi(x)).
\]

On integrating,

\[
\cos \theta_c \mathcal{J}(\mu_{\theta_c}) + \sin \theta_c \mathcal{G}(\mu_{\theta_c}) \geq \cos \theta_c \mathcal{J}(\phi) + \sin \theta_c \mathcal{G}(\phi)
\]

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so that for all $p(x)$ satisfying the side condition (2)

(8) \[ \mathcal{F}(\mu_\theta) \geq \mathcal{F}(p). \]

Thus, the problem is solved if the existence of $\theta_c$ can be demonstrated. Assurance of this is given by the following theorem, and in the course of its proof an admissible $\mu_{\theta_c}(x)$ is constructed:

**Theorem.** If $\mathcal{F}(p)$ and $G(p)$ are defined as in (1) and (2), if the class $\mathcal{P}$ is defined by $A_1$ and $A_2$, and if hypotheses $H_1$, $H_2$, $H_3$, and $H_4$ are all satisfied, then there exists $\theta_c$ in the open interval $(-\pi/2, \pi/2)$ and $\mu_{\theta_c}(x) \in \mathcal{P}$ such that for all $x \in E$

(9) \[ h(\theta_c, x, \mu_{\theta_c}(x)) = \max_{u(x) \leq \mu \leq v(x)} h(\theta, x, \mu) \]

and

(10) \[ G(\mu_{\theta_c}) = c, \]

and, for all $p \in \mathcal{P}$ such that $G(p) = c$,

(11) \[ \mathcal{F}(\mu_{\theta_c}) \geq \mathcal{F}(p). \]

The final assertion (11) has already been proved. In connection with proving the existence of $\theta_c$ we first establish the continuity of $h(\theta, x)$.

Define $F(p, x) = F(v(x), x)$ for $p > v(x)$ and $F(p, x) = F(u(x), x)$ for $p < u(x)$, similarly for $G(p, x)$. $h(\theta, x)$ is then the maximum of $h(\theta, x, \mu)$ over the closed interval $\inf_{x \in E} u(x) \leq \mu \leq \sup_{x \in E} v(x)$. As the upper envelope of the continuous functions $h(\theta, x, \mu)$, $h(\theta, x)$ is lower semi-continuous. To show that $h(\theta, x)$ is also upper semi-continuous consider the set $(\theta, x, \mu)$ for which $h(\theta, x, \mu) \geq \rho$. This set is bounded and closed. Its projection onto the $\theta$, $x$ domain is therefore also closed. This projection is, however, the set of points $(\theta, x)$ for which $h(\theta, x) \geq \rho$. Consequently, $h(\theta, x)$ is also upper semi-continuous.

We now consider for each fixed $\theta$ and $x$ the point set $\{\mu_{\theta}^+(x)\}$ of maximizing $\mu$'s. This set is closed by virtue of the continuity of $h(\theta, x, \mu)$ in $\mu$, and bounded by $u(x)$ and $v(x)$. Therefore, there exist functions $\mu_{\theta}^+(x)$ and $\mu_{\theta}^-(x)$ in $\{\mu_{\theta}^+(x)\}$ satisfying

(12) \[ G(\mu_{\theta}^+(x), x) = \sup_{\alpha} G(\mu_{\theta}^\alpha(x), x), \]

(13) \[ G(\mu_{\theta}^-(x), x) = \inf_{\alpha} G(\mu_{\theta}^\alpha(x), x). \]

We shall now show that $G(\mu_{\theta}^+(x), x)$ is upper semi-continuous and $G(\mu_{\theta}^-(x), x)$ is lower semi-continuous. Let $\{\theta_j, x_j\}$ be any sequence
of points in the $\theta, x$ domain tending to $(\theta_0, x_0)$ for which $\mu_{\theta_j}^+(x_j)$ converges. Then,

$$h(\theta_0, x_0, \lim_{j \to \infty} \mu_{\theta_j}^+(x_j)) = \lim_{j \to \infty} h(\theta_j, x_j, \mu_{\theta_j}^+(x_j)) = h(\theta_0, x_0)$$

so that

$$\lim_{j \to \infty} \mu_{\theta_j}^+(x_j) = \mu_{\theta_0}^+(x_0)$$

for some $\alpha$. Consequently,

$$\lim_{j \to \infty} G(\mu_{\theta_j}^+(x_j), x_j) = G(\lim_{j \to \infty} \mu_{\theta_j}^+(x_j), x_0) = G(\mu_{\theta_0}^+(x_0), x_0) \leq G(\mu_{\theta_0}^+(x_0), x_0).$$

Similarly, $G(\mu_{\theta}^-(x), x)$ is lower semi-continuous.

These functions are therefore summable over $E$, and since $F(\mu_{\theta_j}^+(x), x)$ and $F(\mu_{\theta}^-(x), x)$ can be expressed in terms of the $G$'s together with $h(\theta, x)$, they too are summable over $E$. Thus $\mu_{\theta_j}^+$ and $\mu_{\theta}^-$ are admissible functions.

Furthermore, the integral $G(\mu_{\theta}^+)$ is upper semi-continuous and the integral $G(\mu_{\theta}^-)$ is lower semi-continuous. This follows via Fatou's lemma, viz.

$$\limsup_{\theta \to \theta_0} \int_E G(\mu_{\theta}^+(x), x) dx \leq \int_E \limsup_{\theta \to \theta_0} G(\mu_{\theta}^+(x), x) dx$$

Similarly, $G(\mu_{\theta}^-)$ is lower semi-continuous.

We now define the sets $S^+$ and $S^-$ as follows:

$$S^+ = \{ \theta \mid -\pi/2 < \theta < \pi/2 \text{ and } G(\mu_{\theta}^+) \geq c \},$$

$$S^- = \{ \theta \mid -\pi/2 < \theta < \pi/2 \text{ and } G(\mu_{\theta}^-) \leq c \}.$$

By virtue of the semi-continuity of the integrals these sets are both closed relative to the open interval $(-\pi/2, \pi/2)$. Neither exhausts $(-\pi/2, \pi/2)$. This follows since for all $p \in \mathcal{P}$
\[ G(\mu_{-\pi/2}) \leq G(p) \]

so that

\[ G(\mu_{-\pi/2}) < c. \]

But

\[ \limsup_{\theta \to -\pi/2} G(\mu_\theta^+) \leq G(\mu_{-\pi/2}) < c \]

so that there exists \( \theta \in (-\pi/2, \pi/2) \) such that \( G(\mu_\theta^+) < c \). Thus \( S^+ \) is not equal to \((-\pi/2, \pi/2)\). Similarly \( S^- \) is not equal to \((-\pi/2, \pi/2)\).

Since \((-\pi/2, \pi/2)\) is connected, either

\[ S^+ \cap S^- \neq 0 \]

or

\[ \text{comp } S^+ \cap \text{comp } S^- \neq 0, \]

otherwise \( S^+ \) and \( S^- \) would form a separation. But since \( G(\mu_\theta^+) \geq G(\mu_\theta^-) \), the second alternative is absurd. We therefore pick \( \theta = \theta_e \) in \( S^+ \cap S^- \). Letting

\[ \mu_\lambda(x) = \begin{cases} 
\mu_\theta^+(x) & \text{for } x \leq \lambda, \\
\mu_\theta^-(x) & \text{for } x > \lambda,
\end{cases} \quad \lambda \in E, \]

we consider the integral

\[ G(\mu_\lambda) = \int_{x \leq \lambda, x \in E} G(\mu_\theta^+(x), x) \, dx + \int_{x > \lambda, x \in E} G(\mu_\theta^-(x), x) \, dx. \]

This integral is a continuous function of \( \lambda \) and covers the range

\[ G(\mu_\lambda^-) \leq c \leq G(\mu_\lambda^+). \]

Thus, \( \lambda_e \) exists for which \( G(\mu_{\lambda_e}) = c \). Therefore, \( \mu_{\lambda_e}(x) = \mu_{\theta_e}(x) \) is a solution, and the theorem is proved.

The theorem above provides the basis for a maximization procedure which may be stated as follows:

In order to determine a maximizing function \( p \in P \) for the integral (1) with side condition (2), \( H_1, H_2, H_3, \) and \( H_4 \) being assumed, it is necessary only to maximize the associated integrand

\[ \cos \theta F(p(x), x) + \sin \theta G(p(x), x) \]

over the admissible values of \( p(x) \) for almost all \( x \in E \) and choose \( \theta \) so that condition (2) is satisfied with the corresponding maximizing \( p \).

Through establishing the existence of \( \theta_e \) the theorem asserts that
the above procedure will always yield a solution. Consequently,

**Corollary 1.** The problem has at least one solution.

Furthermore,

**Corollary 2.** Every solution $p^*$ conforms to the maximizing procedure—and with common $\theta$.

Assuming the denial, we put $\theta = \theta_c$ and have

$$\cos \theta_c F(p^*(x), x) + \sin \theta_c G(p^*(x), x) < \cos \theta_c F(\mu_{\theta_c}(x), x) + \sin \theta_c G(\mu_{\theta_c}(x), x)$$

on some subset of $E$ of measure $> 0$. An integration leads to the contradiction.

Hypothesis $H_2$ may be somewhat revised, as indicated by

**Corollary 3.** Bound free problems (i.e., $u(x) = -\infty$ and $v(x) = +\infty$), as well as problems with discontinuous bound functions, conform to the maximization procedure provided it is known that a bounded solution exists.

**Corollary 4.** If the bound free problem has a solution $p'$, then $p'$ conforms to the maximization procedure on any compact subset $E' \subseteq E$ on which $p'$ is bounded.

**Remarks.** Regarding the admissibility conditions for the class $\mathcal{P}$, the first, $A_1$, was imposed in order to bring problems demanding such bounds on $p$ into the domain of the maximizing principle. It may be relaxed under certain circumstances depending on the form of $F$ and $G$ (for example, see Corollaries 3 and 4). $A_2$ is obviously needed. A third, $A_3$, imposing measurability on all $p \in \mathcal{P}$ might possibly be included if a suitable rule could be set down (such as taking sup or inf perhaps) for selecting the $\mu^+(x)$ and $\mu^-(x)$ at each point.

Although hypotheses $H_1$ and $H_3$ are about as weak as they need be for many practical problems, they might still be further weakened. This has not been investigated. On the other hand, $H_4$ is necessary, for, consider the case $c = \text{Max} \ G(p)$. Here a permissible $\theta$ is $\pi/2$, and the maximizing procedure yields all $p \in \mathcal{P}$ satisfying $G(p) = c$. Some of these may yield $\mathcal{J}(p)$ larger than others.

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