TOPOLOGICAL DEGREE OF SOME MAPPINGS

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1. In previous papers [2; 3], the problem of proving an existence theorem for a certain functional equation was reduced to that of computing the topological degree of a mapping in Euclidean n-space defined by homogeneous polynomials or infinite series. The complex case of the latter problem was solved in [4]. Since the problem is analogous to that of studying the roots of a polynomial equation, we would expect the real case to be more complicated. Here we obtain a result that is an analogue of the theorem that a real polynomial equation of odd degree has at least one real root. Also we describe the solution for the case $n=2$ if the mapping is defined by homogeneous polynomials.

2. We consider the mapping of real Euclidean $n$-space $\mathbb{R}^n$ into itself,

$$M : (x_1, \ldots, x_n) \mapsto (x'_1, \ldots, x'_n)$$

defined by

$$x'_i = \sum_{m=2}^{\infty} \sum_{j} a_{j_1 \ldots j_n} x_1^{j_1} \cdots x_n^{j_n} \quad (i = 1, \ldots, n)$$

where $\sum_j$ denotes a summation taken over all sets of non-negative integers $j_1, \ldots, j_n$ such that $\sum_{q=1}^{n} j_q = m$. The problem is to determine the topological degree at 0 of $M$. Let $\mathcal{M}$ be the mapping of complex Euclidean $\mathbb{R}^n$ into itself that corresponds to $M$, i.e., $\mathcal{M}$ is defined by

$$z'_i = \sum_{m=2}^{\infty} \sum_{j} a_{j_1 \ldots j_n} z_1^{j_1} \cdots z_n^{j_n} \quad (i = 1, \ldots, n).$$

Let $S$ be a sphere in $\mathbb{R}^n$ with center 0 such that $d[M, S, 0]$, the topological degree at 0 of $M$ relative to $S$, is defined, and let $\mathcal{S}$ be the corresponding sphere in $\mathbb{R}^n$, i.e., a sphere in $\mathbb{R}^n$ with center 0 and radius equal to the radius of $S$. Suppose first that $d[\mathcal{M}, \mathcal{S}, 0]$ is defined, i.e., suppose $\mathcal{M} \neq 0$ on the surface of $\mathcal{S}$. We prove:

(1) \[ |d[M, S, 0]| \leq d[\mathcal{M}, \mathcal{S}, 0] \]

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(2) \[ d[M, S, 0] \equiv d[\mathcal{M}, S, 0] \pmod{2}. \]

By Lemma 3.1 of [3] (which is a special case of a theorem due to Sard [6]) and the fundamental property of topological degree [1, Deformationssatz, p. 424] there is a real point \( p \) near 0 such that

(a) \[ d[M, S, 0] = d[M, S, p], \]
(b) \[ d[\mathcal{M}, S, 0] = d[\mathcal{M}, S, p], \]
(c) The set \( M^{-1}(p) \) is finite.

Let \( q_1, \ldots, q_r \) be the elements of \( M^{-1}(p) \) and let \( J \) be the Jacobian of \( M \). Then

\[ d[M, S, p] = \sum_{i=1}^{r} \text{sign}(J(q_i)). \]

Since \( J(q_i) \neq 0 \) for \( i = 1, \ldots, n \), the points \( q_1, \ldots, q_r \) are isolated points in the set \( \mathcal{M}^{-1}(p) \). As proved in [4], the topological index of \( M \) at each \( q_i \) is +1. Hence from the properties of topological degree [1, Satz 11, p. 472], it follows that

\[ d[\mathcal{M}, S, p] = d[\mathcal{M}, S - S, p] + r. \]

But since the coefficients in \( \mathcal{M} \) are real, it follows easily that \( d(\mathcal{M}, S - S, p) \) is a positive, even number. (This is proved in [5].) Since

\[ d[M, S, p] \equiv r \pmod{2}, \]

the proof is complete.

Now suppose \( \mathcal{M} = 0 \) at some point on the surface of \( S \). We assume that the coefficients in the series that define \( M \) and \( \mathcal{M} \) may be varied slightly so that the following result is obtained: the mappings \( M_1 \) and \( \mathcal{M}_1 \) in \( \mathbb{R}^n \) and \( \mathbb{R}^n \), respectively, defined by the new series are such that

(\( \alpha \)) \[ d[M, S, 0] = d[M_1, S, 0], \]
(\( \beta \)) \( \mathcal{M}_1 \) is different from zero on \( S \), i.e., \( d[\mathcal{M}_1, S, 0] \) is defined.

Then applying the preceding argument to \( M_1 \) and \( \mathcal{M}_1 \), we obtain:

(3) \[ |d[M, S, 0]| = |d[M_1, S, 0]| \leq d[\mathcal{M}_1, S, 0] \]

and

(4) \[ d[M, S, 0] = d[M_1, S, 0] = d[\mathcal{M}_1, S, 0] \pmod{2}. \]

In particular, if \( M \) is defined by homogeneous polynomials, i.e.,

\[ x'_i = P_k(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \]
where $P_{ki}$ is homogeneous of degree $k_i$ in $x_1, \ldots, x_n$, then

\begin{equation}
|d[M, S, 0]| \leq \prod_{i=1}^{n} k_i
\end{equation}

and

\begin{equation}
d[M, S, 0] \equiv \prod_{i=1}^{n} k_i \pmod{2}.
\end{equation}

This follows from the preceding paragraphs, and the fact, proved in [4], that the topological index of the mapping in complex Euclidean $n$-space defined by

\[ z_i' = P_{ki}(z_1, \ldots, z_n) \quad (i = 1, \ldots, n) \]

is $\prod_{i=1}^{n} k_i$.

Results (5) and (6) may also be obtained by using Bezout's Theorem. This was pointed out to me by R. Brauer before the proof given here was obtained.

3. If $n = 2$ and the mapping is defined by homogeneous polynomials, a solution of the problem can easily be given. First by varying the coefficients slightly (so slightly that the topological degree relative to the unit circle is not affected) we obtain a mapping $M$ defined by

\begin{align*}
x_1' = P_{k_1}(x_1, x_2) &= C_1 \prod_{i=1}^{n} (x_1 - \alpha_i x_2)^{\nu_i}, \\
x_2' = P_{k_2}(x_1, x_2) &= C_2 \prod_{j=1}^{m} (x_1 - \beta_j x_2)^{\eta_j},
\end{align*}

where $C_1, C_2$ are real constants. Since $C_1$ and $C_2$ affect only the sign of the topological degree, we may disregard them. The topological degree can be determined by investigating the changes of sign of $P_{k_1}$ and $P_{k_2}$ as $(x_1, x_2)$ varies over the boundary of the unit circle. Consequently we may disregard factors $(x_1 - \alpha_i x_2)$ or $(x_1 - \beta_j x_2)$ which appear with even exponents or in which $\alpha_i$ and $\beta_j$ are complex since none of these contributes to a change of sign of $P_{k_1}$ or $P_{k_2}$. So we are left with real factors all having exponent one.

Now if there is a pair $\alpha_i, \alpha_{i+1}$ ($\alpha_i < \alpha_{i+1}$) such that no $\beta_j$ lies between then (i.e., there is no $\beta_j$ such that $\alpha_i < \beta_j < \alpha_{i+1}$) then the factors $(x_1 - \alpha_1 x_2)$ and $(x_1 - \alpha_{i+1} x_2)$ may be disregarded since they contribute no significant change of sign to $P_{k_1}$. Similarly for pairs...
Finally if $\alpha_r$ and $\alpha_s$ are the smallest and largest of the array of numbers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$, then the factors $(x_1 - \alpha_r x_2)$ and $(x_1 - \alpha_s x_2)$ may be disregarded. Similarly, $(x_1 - \beta_r x_2)$ and $(x_1 - \beta_s x_2)$ may be disregarded if $\beta_r$ and $\beta_s$ are the smallest and largest.

Now if there are no remaining factors in $P_{k_1}$ or in $P_{k_2}$, the topological degree is zero. Otherwise there remain factors containing numbers $\alpha_1, \ldots, \alpha_w$ and $\beta_1, \ldots, \beta_w$ where all the $\alpha$'s and $\beta$'s are distinct and, if the subscript labelling is according to magnitude,

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_w < \beta_w$$

or

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_w < \alpha_w.$$ 

In the first case the degree is $w$; in the second case $-w$.

**Bibliography**


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