CONCERNING INTEGRALS

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1. Introduction. R. E. Lane [2] has given the following definition of an integral on the interval \([a, b]\) of the function \(f\) with respect to the function \(g\). If \(D\) is an ordered subdivision \(\{x_i\}_{i=0}^{n+1}\) of the interval \([a, b]\), \(\sum_D(f, g)\) denotes the sum

\[
\sum_{i=1}^{n+1} \frac{1}{2} [f(x_i) + f(x_{i-1})] [g(x_i) - g(x_{i-1})].
\]

The statement that \(f\) is \(g\)-integrable on \([a, b]\) means that there exists a number \(J\) such that for each positive number \(\epsilon\) there is an ordered subdivision \(D\) of \([a, b]\), such that for every refinement \(D'\) of \(D\), \(|J - \sum_{D'}(f, g)| < \epsilon\). The number \(J\) is the integral on \([a, b]\) of \(f\) with respect to \(g\), and is denoted by \(\int_a^b f dg\). This integral generalizes the Stieltjes integral and has many of its properties, e.g., is an additive function of intervals and a bilinear function of \((f, g)\); if \(f\) is \(g\)-integrable on \([a, b]\), then \(g\) is \(f\)-integrable on \([a, b]\) and \(\int_a^b gdf = \int_a^b f dg\). If \(g \in BV[a, b]\) and \(f\) has only discontinuities of the first kind in \([a, b]\), then \(f\) is \(g\)-integrable on \([a, b]\) and, in particular, if \(g\) is a simple step-function, then

\[
\int_a^b f dg = \sum_{a < x \leq b} \frac{f(x) + f(x -)}{2} [g(x) - g(x -)] + \sum_{a \leq x < b} \frac{f(x +) + f(x)}{2} [g(x +) - g(x)].
\]

Suppose \(g\) is a nondecreasing function and \(\{f_n\}_{n=1}^\infty\) a uniformly bounded sequence of simple step-functions converging to a function \(f\) in \([a, b] - S\), where \(S\) is a subset of \([a, b]\) of “\(g\)-length 0.” It is to be expected that the methods of F. Riesz [3] can be applied to the sequence \(\{\int_a^b f_n dg\}_{n=1}^\infty\), and its limit used to define a Riesz type integral of \(f\) with respect to \(g\) on \([a, b]\).

In this paper we have reached the desired result, but by methods which are in some way even more elementary than those of Riesz.

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1 The statement that \(g\) is a simple step-function means that \(g\) is a function on the set of all numbers and if \([a, b]\) is an interval, then there exist a subdivision \(a = x_0 < x_1 < \ldots < x_n = b\) and a sequence \(\{k_p\}_{p=1}^\infty\) of numbers, such that \(g(x) = k_p\) if \(k_{p-1} < x < k_p\).
We depend upon approximation to the nondecreasing function $g$ by step-functions, and essentially reduce the question of measure of $S$ to that of the measure of a finite subset of $S$.

2. **Outer $g$-length of a number set.** Throughout this paper, $g$ denotes a nondecreasing function on the set of all numbers.

If $S$ is a number set, then the statement that $l_g S$ is the outer $g$-length of $S$ means that $l_g S$ is the largest number $k$ such that if $G$ is a countable collection of segments covering $S$, then $k \leq \sum [g(q) - g(p)]$, the sum being taken over every segment $(p, q)$ in $G$.

We state here, without proof, some elementary properties of outer $g$-length.

(i) The outer $g$-length of the interval $[a, b]$ is $g(b+) - g(a-)$.  
(ii) The outer $g$-length of the segment $(a, b)$ is $g(b- - g(a+)$.
(iii) If $Q$ is an open and bounded number set and $\epsilon$ a positive number, then there exists a finite collection $G$ of mutually exclusive intervals, such that if $G^*$ is a subset of $Q$, then $l_g[a, b] = g(b) - g(a)$ for each interval $[a, b]$ in $G$, and $0 \leq l_g Q - l_g G^* < \epsilon$.

(iv) If $S$ and $T$ are bounded and mutually exclusive number sets and $h$ is a nondecreasing simple step-function, then $l_h(S+T) = l_h S + l_h T$.

(v) If each of $S$ and $T$ is a bounded number set, then $l_g(S+T) \leq l_g S + l_g T$.

**Theorem A.** If $S$ is a bounded number set, each of $\epsilon$ and $\delta$ a positive number and $l_g S \geq \delta$, then there exists a nondecreasing simple step-function $h$, such that, for, every number $x$, $|h(x) - g(x)| < \epsilon$ and $l_h S \geq \delta$.

**Proof.** Suppose $[a, b]$ is an interval containing $S$. There exists an ordered subdivision $\{x_i\}_{i=0}^{n+1}$ of $[a, b]$ such that if $x \in (x_{i-1}, x_i)$ then $|g(x) - g(x_{i-1})| < \epsilon/2$. Suppose $\{y_i\}_{i=1}^{n+1}$ is a sequence of numbers such that $y_i \in (x_{i-1}, x_i)$ and $y_i \in S$ if $(x_{i-1}, x_i)$ contains a number belonging to $S$. There exists a number $a'$ less than $a$, such that if $x \in (a', a)$ then $|g(x) - g(a-)| < \epsilon/2$ and a number $b'$ greater than $b$, such that if $x \in (b, b')$ then $|g(x) - g(b+)| < \epsilon/2$.

There exist a simple step-function $h_1$ on the set of all numbers less than or equal to $a'$, such that if $x \leq a'$ then $|h_1(x) - g(x)| < \epsilon/2$, $h_1(a') = g(a')$, and $h_1$ is nondecreasing, and a simple step-function $h_2$ on the set of all numbers greater than or equal to $b'$, such that if $x \geq b'$ then $|h_2(x) - g(x)| < \epsilon/2$, $h_2(b') = g(b')$, and $h_2$ is nondecreasing (cf. [2]).

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2 If $G$ is a collection of sets, then $G^*$ denotes the set that is the logical sum of the sets in $G$.
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h denotes the simple step-function defined as follows: if \( x \leq a' \), \( h(x) = h_1(x) \); if \( x \geq b' \), \( h(x) = h_2(x) \); if \( a' < x < a \), \( h(x) = g(a' +) \); if \( b < x < b' \), \( h(x) = g(b' -) \), \( h(x_i) = g(x_i) \) if \( i = 0, 1, \ldots, n + 1 \); if \( y_i \in S \) and \( x_{i-1} < x < x_i \), \( h(x) = g(y_i) \); and if \( y_i \in S, \ h(x) = g(x_{i-1}+) \) or \( h(x) = g(x_i-) \), according as \( x_{i-1} < x < y_i \) or \( y_i < x < x_i \), respectively.

h is a nondecreasing simple step-function, and if \( x \) is a number, then \( |h(x) - g(x)| < \epsilon \).

Suppose \( l_h S < \delta \).

If \( S \) is a subset of \( \{x_i\}_{i=0}^{n+1} \) then, inasmuch as \( g(x_i+) - g(x_i-) \leq h(x_i+) - h(x_i-) \), we see that \( l_o S \leq l_h S < \delta \), contrary to the hypothesis of the theorem. Therefore, there is a subset of \( S \) in one of the segments \( (x_{i-1}, x_i) \). Suppose \( M \) denotes the collection of these segments containing subsets of \( S \), and \( S_1 \) is the common part of \( S \) and \( M^* \). If one of the numbers \( x_i \) belongs to \( S \), denote by \( S_2 \) the common part of \( S \) and \( \{x_i\}_{i=0}^{n+1} \) and suppose \( k = l_h S_2 \). If \( S = S_1 \) then \( k = 0 \).

Suppose \( \sum_{(p, q) \in M} [g(q) - g(p)] < l_o S - k \). Then there exists a positive number \( t \) such that \( \sum_{(p, q) \in M} [g(q) - g(p)] + k + t < l_o S \). If \( S = S_1 \), so that \( k = 0 \), this implies \( l_o S < l_o S \), an absurdity. If \( S \neq S_1 \), there is a finite collection \( H \) of segments covering \( S_2 \) such that \( \sum_{(p, q) \in H} [g(q) - g(p)] < k + t \), and therefore

\( l_o S \leq \sum_{(p, q) \in M + H} [g(q) - g(p)] < l_o S \),

an absurdity. Consequently, we see that

\[ k + \sum_{(p, q) \in M} [g(q) - g(p)] \geq l_o S \geq \delta. \]

If \( S = S_1 \), this states that \( l_h S \geq \delta \).

If \( S \neq S_1 \), this states that \( l_h S_2 + l_h S_1 \geq \delta \) so that, by (iv), \( l_h S \geq \delta \).

Thus, the supposition \( l_h S < \delta \) is false, and Theorem A is established.

3. Sequences of simple step-functions. The following theorem is along the lines of a theorem of Egoroff [1].

Theorem B. If \( S \) is a proper subset of the interval \([a, b]\), \( l_o S = 0 \), \( \{h_n\}_{n=1}^{\infty} \) is a sequence of simple step-functions such that, for each number \( x \) in \([a, b] - S\), \( h_n(x) \to 0 \) as \( n \to \infty \), and each of \( \epsilon \) and \( \delta \) is a positive number, then there exists a subset \( T \) of \([a, b] - S\) and a positive integer \( N \) such that, for each \( x \) in \( T \) and each integer \( n \) greater than \( N \), \( |h_n(x)| < \epsilon \) and \( l_o T \geq l_o [a, b] - \delta \).

Proof. Suppose there exists a positive number \( \epsilon \) and a positive number \( \delta \), such that if \( N \) is a positive integer, \( T \) a subset of \([a, b] - S\) and, for each integer \( n \) greater than \( N \) and each \( x \) in \( T \), \( |h_n(x)| < \epsilon \), then \( l_o T < l_o [a, b] - \delta \).

If \( \epsilon \) is a positive integer, then there exists a number \( x \) in \([a, b] - S\)
such that for some integer $m$ greater than $n$, $|h_m(x)| \geq \varepsilon$. Otherwise, the set $[a, b] - S$ is a set $T$ for which the above supposition is violated. For each positive integer $n$, $U_n$ denotes the set such that $x \in U_n$ only if $x \in [a, b] - S$ and $|h_m(x)| \geq \varepsilon$ for some integer $m$ greater than $n$. We see that $U_{n+1}$ is a subset of $U_n$. We shall prove that there exists a nonempty and closed number set $C_n$ such that $C_n$ is a subset of $U_n$ and $C_{n+1}$ is a subset of $C_n$. If $y \in C_n$ for $n = 1, 2, 3, \ldots$, then $y \in [a, b] - S$ and $h_n(y) \not\to 0$ as $n \to \infty$, contrary to the hypothesis of the theorem. This contradiction will show that our supposition is false and the theorem will then be established.

If $r_n = l \log U_n$, then $\{r_n\}_{n=1}^\infty$ is a nonincreasing sequence with a nonnegative limit $r$. If $r = 0$ and $k$ is a positive integer such that $r_k < \delta$, the set $[a, b] - [S + U_k]$ is a set $T$ for which our supposition is violated. Consequently, $r > 0$.

$H$ denotes the set such that $x \in H$ only if $x$ is $a$, $b$, or a number in $[a, b]$ where, for some positive integer $n$, $h_n$ is not continuous. If $H$ is finite, $K = H$; if $H$ is infinite, $K$ is a finite subset of $H$ containing $a$ and $b$, such that $\sum_{x \in H - K} [g(x+) - g(x-)] < r\theta$, where $\theta$ is a positive number less than $1/4$. If there is a number $y$ such that $y \in K$ and $y \in U_n$ for $n = 1, 2, 3, \ldots$, we take $C_n = (y)$ for $n = 1, 2, 3, \ldots$.

Suppose there is a positive integer $k$ such that $K$ and $U_k$ have no common part. If $n$ is an integer greater than $k$, $U_n$ is not a subset of $H$; $U'_n$ denotes $U_n - H \cdot U_n$; $Q_n$ denotes the set such that $x \in Q_n$ if and only if $x$ is in the segment $(a, b)$ and there is an integer $m$ greater than $n$ such that $h_m$ is continuous at $x$ and $|h_m(x)| \geq \varepsilon$. $Q_n$ is open, $Q_{n+1} \subset Q_n$, and $U'_n = Q_n - (H+S) \cdot Q_n$; if $x \in K$ and $x \in S$, then $x \in Q_n$.

There exists a finite collection $G_1$ of mutually exclusive intervals such that if $[p, q] \in G_1$ then $g$ is continuous at $p$ and at $q$, $G_1^* \subset Q_{k+1}$, and $0 \leq l \log Q_{k+1} - l \log G_1^* < r\theta$, so that $l \log G_1^* > r - r\theta$. If $i$ is a positive integer less than $k + 2$, $C_i$ denotes the closed set $G_i^*$. If $i$ is an integer greater than 1, then $l \log (Q_{k+1} \cdot G_1^*) \geq r - r\theta$.

For each integer $i$ greater than 1, there exists a finite collection $G_i$ of mutually exclusive intervals such that if $[p, q] \in G_i$, then $g$ is continuous at $p$ and at $q$, $G_i^* \subset Q_{k+1} \cdot G_i^* - 1$, $0 \leq l \log (Q_{k+1} \cdot G_i^*) - l \log G_i^* < r\theta$ and, for each integer $j$ greater than $i$, $l \log (Q_{k+j} \cdot G_i^*) \geq r - r\theta - \cdots - r\theta^j$. For each positive integer $i$, $C_{i+1}^*$ denotes the closed set $G_i^*$. Suppose $Q$ is an open set covering $(H+S) \cdot Q_k$ such that $l \log Q < r\theta$. Since $l \log C_{i+1}^* > r(1 - 2\theta)/(1 - \theta) > r\theta$, then $C_i'$ is not a subset of $Q$ ($n = 1, 2, 3, \ldots$). If $C_n$ is the closed set $C_n' - C_n' \cdot Q$, then $U_n \supset C_n \supset C_{n+1}$, and Theorem $B$ is established.

4. $g$-summable functions. In this section we consider functions on an interval $[a, b]$ and suppose the nondecreasing function $g$ is such
that $g(x) = g(a)$ for $x < a$ and $g(x) = g(b)$ for $x > b$.

**Theorem C.** If $S$ is a proper subset of the interval $[a, b]$, $l_0S = 0$, \( \{f_n\}_{n=1}^\infty \) a sequence of simple step-functions, uniformly bounded on $[a, b]$, which converges to 0 on $[a, b] - S$, $f_n(x+) \to 0$ as $n \to \infty$ if $a < x < b$ and $g(x+) > g(x)$ and $f_n(x-) \to 0$ as $n \to \infty$ if $a < x \leq b$ and $g(x) > g(x-)$, then

$$\int_a^b f_n \, dg \to 0 \quad \text{as } n \to \infty.$$ 

**Proof.** Suppose $M$ is a number such that if $x \in [a, b]$ and $n$ is a positive integer, then $|f_n(x)| < M$. Suppose $\epsilon$ is a positive number and $\epsilon_1 = \epsilon / \{4 + 8[g(b) - g(a)]\}$ and $\delta_1 = \epsilon / 2M$.

There exists a subset $T_1$ of $[a, b] - S$ and a positive integer $N_1$ such that, if $n > N_1$ and $x \in T_1$, then $|f_n(x)| < \epsilon_1$ and $l_0 T_1 \geq l_0 [a, b] - (\delta_1 / 2)$. Suppose $H$ is the set such that $x \in H$ if and only if $x \in [a, b]$ and $g$ is not continuous at $x$ or, for some positive integer $n$, $f_n$ is not continuous at $x$. There exists a finite subset $K$ of $H$ such that

$$\sum_{x \in H - K} [g(x+) - g(x-)] < \delta_1 / 2.$$

If $T = T_1 - (H - K) \cdot T_1$, then $l_0 T + l_0 [a, b] - \delta_1$.

There exists a sequence $\{h_n\}_{n=0}^\infty$ of nondecreasing simple step-functions such that $h_n(a) = g(a)$, $h_n(b) = g(b)$, $l_0 T \geq l_0 [a, b] - \delta_1$ or $l_0 \{[a, b] - T\} \leq \delta_1$ and, for each number $x$ in $[a, b]$, $|g(x) - h_n(x)| < 1/n$.

Now, if each of $m$ and $n$ is a positive integer (cf. (1.1))

$$\int_a^b f_m \, dh_n = \sum_{x \in T \cdot (a, b)} \frac{f_m(x) + f_m(x-)}{2} \left[ h_n(x) - h_n(x-) \right]$$

$$+ \sum_{x \in T \cdot (a, b)} \frac{f_m(x+) + f_m(x)}{2} \left[ h_n(x+) - h_n(x) \right] + \sum_3,$$

where $| \sum_3 | \leq M \delta_1 = \epsilon / 2$, $\sum_3$ being a sum of like terms taken for $x \in [a, b] - T$. If we consider separately those terms for which $x \in K \cdot T$ and for which $x \in T - K \cdot T$, we see that there exists a positive integer $N$ such that if $m, n > N$, then $|\int_a^b f_m \, dh_n| < \epsilon$.

Now, $\int_a^b f_m \, dg = \int_a^b f_m \, dh_n + \int_a^b f_m \, (g - h_n)$ so that, if we use integration by parts, and $m, n > N$,

$$|\int_a^b f_m \, dg| \leq |\int_a^b f_m \, dh_n| + |\int_a^b (g - h_n) \, df_m| \leq \epsilon + \frac{1}{n} V_a f_m$$

or $\int_a^b f_m \, dg \to 0$ as $m \to \infty$.}

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This completes the proof of Theorem C.

The statement that the function \( f \) is \( g \)-summable on \( [a, b] \) means that there exists a sequence \( \{f_m\}_{m=1}^{\infty} \) of simple step-functions, uniformly bounded on \( [a, b] \), such that \( f_m(x) \to f(x) \) as \( m \to \infty \) for every number \( x \) in \( [a, b] \) or in \( [a, b] - S \), where \( S \) is a subset of \( [a, b] \) of outer \( g \)-length 0, and \( f_m(x-) \to f(x-) \) as \( m \to \infty \) if \( a < x \leq b \) and \( g(x) > g(x-) \), and \( f_m(x+) \to f(x+) \) as \( m \to \infty \) if \( a \leq x < b \) and \( g(x+) > g(x) \).

We see by Theorem C that if \( f \) is \( g \)-summable on \( [a, b] \), then there exists a number \( J \) such that if \( \{f_m\}_{m=1}^{\infty} \) is any sequence of simple step-functions having the above properties:

\[
\int_{a}^{b} f_m \, dg \to J \quad \text{as } m \to \infty.
\]

We define the number \( J \) to be the integral \( \int_{a}^{b} f \, dg \) on \( [a, b] \) of \( f \) with respect to \( g \).

It is easy to show that if \( \{f_m\}_{m=1}^{\infty} \) is a uniformly bounded sequence of \( g \)-summable functions converging in the manner described in the above definition to a function \( f \), then \( f \) is \( g \)-summable and \( \int_{a}^{b} f_m \, dg \to \int_{a}^{b} f \, dg \) as \( m \to \infty \).

Remark added in proof. My attention has been called to the fact that the definition I accredited to Lane was given by H. L. Smith, On the existence of the Stieltjes integral (Trans. Amer. Math. Soc. vol. 27 (1925) pp. 491–495).

Bibliography


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