NOTE ON A THEOREM OF OSTROWSKI

ALFRED HUBER

Generalizing a result of W. F. Osgood [1], A. M. Ostrowski [2] proved the following theorem: Let \( f(t, x) \) be a continuous function of the point \( (t, x) \) for \( a < x < b \) and \( t \geq T \) and suppose that we have
\[
\lim_{t \to \infty} f(t, x) = f(x),
\]
where \( f(x) \) is also continuous in \( (a, b) \); then for any \( \epsilon > 0 \) there exists a subinterval \( J \) of \( (a, b) \) and a number \( T_0 \) such that we have for \( x \in J, t \geq T_0 \):
\[
| f(t, x) - f(x) | < \epsilon.
\]

It is the purpose of this note to show that even the following statement is true:

**Theorem.** Let the function \( f(t, x) \) be defined for \( a < x < b \), \( t \) varying on a set \( E \) of real numbers, which is assumed to be unbounded from above, but otherwise arbitrary; let \( f(t, x) \) be continuous in \( x \) for fixed \( t \) and suppose that we have
\[
\lim_{t \to \infty} f(t, x) = f(x),
\]
where \( f(x) \) is also continuous in \( (a, b) \). Then for any \( \epsilon > 0 \) there exists a subinterval \( J \) of \( (a, b) \) and number \( N \) such that we have for all \( x \in J, t \in E_N \), where \( E_N = E \cap \{ t \geq N \} \),
\[
| f(t, x) - f(x) | < \epsilon.
\]

Our proof is indirect. The opposite statement can be formulated as follows:

There exists a number \( \epsilon_0 > 0 \) with the following property: Given any subinterval \( J \) of \( (a, b) \) and an arbitrary number \( N \), there always exists a point \( x' \in J \) and a number \( t' \in E_N \) such that
\[
| f(t', x') - f(x') | > \epsilon_0
\]
holds. (Clearly we may require \( x' \) to be an interior point of \( J \).)

Let us now consider an arbitrary closed subinterval \( (a_1, b_1) \) of \( (a, b) \). There exists a point \( x_1 \), interior to \( (a_1, b_1) \), and a number \( t_1 \in E_1 \)

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such that

\[
| f(t_1, x_1) - f(x_1) | > \epsilon_0
\]

is fulfilled. Since both \( f(x) \) and \( f(t_1, x) \) are continuous functions of \( x \), (1) remains valid in a sufficiently small neighborhood \( U_1 \) of \( x_1 \) if we keep \( t_1 \) fixed. Let \( (a_2, b_2) \) denote a closed interval, contained in both \( U_1 \) and \( (a_1, b_1) \).

There exists a point \( x_2 \), interior to \( (a_2, b_2) \), and a number \( t_2 \in E_2 \) such that

\[
| f(t_2, x_2) - f(x_2) | > \epsilon_0
\]

holds. (2) remains valid in a neighborhood \( U_2 \) of \( x_2 \). Let \( (a_3, b_3) \) denote a closed interval, contained in both \( U_2 \) and \( (a_2, b_2) \). And so forth.

We thus obtain a nested sequence of closed intervals. They have—at least—one point \( x^* \) in common. We have

\[
| f(t_n, x^*) - f(x^*) | > \epsilon_0 \quad (n = 1, 2, 3, \ldots).
\]

Since \( t_n \to \infty \) for \( n \to \infty \), (3) is a contradiction to the hypothesis that \( \lim_{t \to \infty} f(t, x^*) = f(x^*) \). This proves the theorem.

The proof is still valid, if—as pointed out by J. B. Diaz—the continuity of \( f(t, x) \) and \( f(x) \) is replaced by the following weaker condition: \( |f(t, x) - f(x)| \) is lower semicontinuous in \( x \) for any fixed \( t \).

The case of a finite limiting point for \( t \) can be treated in an analogous way.

REFERENCES


University of Maryland