1. Introduction. Let \( f \) be a mapping of a space \( A \) onto a space \( B \). Let \( N(y, f, A, B) \), \( y \in B \), be the number of points in \( f^{-1}(y) \). Let \( k \) be a positive integer. According to G. T. Whyburn [2], \( f \) is said to be \( k \)-fold irreducible provided that for \( y \in B \), \( N(y, f, A, B) \geq k \), and if \( F \) is a closed proper subset of \( A \), then for some \( z \in B \), \( N(z, f|F, F, B) < k \). It should be noted that if \( f \) is 1-fold irreducible, then \( f \) is strongly irreducible in accordance with the terminology of [3] or irreducible in accordance with [2].

In this paper there is introduced the notion of a \( k \)-fold irreducible decomposition of a space relative to a mapping as follows.

Definition. Let \( f \) be a mapping defined on a compact space \( A \) onto a space \( B \). \( A \) is said to possess a \( k \)-fold irreducible decomposition relative to \( f \) provided that there exists a decomposition \( A = \bigcup_{i=1}^{n} A_i \) where each \( A_i \) is a nonempty closed subset of \( A \) and the decomposition satisfies the following conditions:

(i) \( A_i^0 \), the interior (rel. to \( A \)) of \( A_i \), is dense in \( A_i \).
(ii) \( A_i^0 \cdot A_j^0 = 0 \), \( i \neq j \).
(iii) \( f(A_i) = B \) and \( f|A_i \) is an irreducible mapping.

We further define \( D(k, f, A, B) \) to be the set of all \( x \in A \) for which \( N(f(x), f, A, B) = k \). When there is no chance of confusion we shall write \( D(k, f, A, B) \) as \( D(k, f, A) \).

In 1.1 through 1.5 some definitions and results of G. T. Whyburn which will be needed are listed. In the remaining part of §1, we obtain results concerning \( k \)-fold irreducible decomposition of a space relative to certain types of mappings. One of the principal results (1.13) states that if \( f \) is a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum, then the set of points \( D \) at which \( f \) is exactly \( k \)-to-one is dense in \( A \) if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \). A corollary of this theorem is that for the same hypothesis as the theorem, if for each \( x \in A \), \( f^{-1}f(x) \) consists of at least \( k \) points, then \( f \) is \( k \)-fold irreducible if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \).
In §2, some examples of $k$-fold irreducible decomposition are given. It is shown that if $f$ is a quasi-monotone mapping defined on a boundary curve $A$ such that for each $x \in A$, $f^{-1}(f(x))$ consists of at least $k$ points, then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one. Also, if $f$ is a quasi-monotone mapping defined on a boundary curve which has a $k$-fold irreducible decomposition relative to $f$, then $f$ is a light open mapping. Finally, it is shown that for an open-mapping $f$ defined on a compact 2-manifold $A$, $A$ has a $k$-fold irreducible decomposition relative to $f$ if and only if the degree of $f$, as defined in [3], is $k$.

For a general reference to topological terms and results used in this paper see [3]. All spaces are assumed to be metric.

1. For any set $A$ in a metric space and any positive $k$, define:
   \[ e_k(A) = \text{g.l.b.} \left( \max_{i=1}^{k} \delta(A_i) \right) \text{ for all decompositions } A = \sum_{i=1}^{k} A_i \text{ of } A \text{ into } k \text{ nonempty subsets } A_i. \]
   Here $\delta$ stands for diameter.

2. For $f$, a mapping defined on a metric space $X$, define
   \[ e_k(x) = e_k(f^{-1}(f(x))), \quad x \in X. \]

3. If the mapping $f$ defined on $X$ generates an upper-semi-continuous decomposition of $X$, the function $e_k(x)$ is upper-semi-continuous.

4. Let $f$ be a mapping defined on a compact space $A$. Suppose for each $x \in A$, $N(f(x), f, A) \leq k$. Then $f$ is $k$-fold irreducible if and only if the set of points $x \in A$ such that $N(f(x), f, A) = k$ is dense in $A$ (i.e. $D(k, f, A)$).

Corollary 1. $f$ is irreducible if and only if the set $D$ of all points $x \in A$ with $x = f^{-1}(f(x))$ is dense in $A$.

Corollary 2. If $f$ is open, then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one.

Corollary 3. If $f$ is open, then $f$ is irreducible if and only if $f$ is a homeomorphism.

5. Let $f$ be a mapping defined on a compact space $A$ onto a space $B$. Then there exists a compact set $A' \subset A$ such that $f(A') = B$ and $f|A'$ is an irreducible mapping.

6. Theorem. Let $f$ be a mapping defined on a compact space $A$ onto a space $B$. If $f$ possesses a $k$-fold irreducible decomposition relative to $f$, then $D(k, f, A, B)$ is dense in $A$.

Proof. We first prove the following.
Let $U$ be an open set in $A$ and let $e > 0$. Then there exists an open set $Q$ in $U$ such that (i) if $q \in Q$, then $f^{-1}(q)$ contains at least $k$ points and (ii) for each $q \in Q$, $e_k(q) < e$, where $e_k(q)$ is as defined in 1.1.

Toward the end of proving this statement, let $A = \sum_{i=1}^{k} A_i$ be a $k$-fold irreducible decomposition relative to $f$ and we suppose further that this decomposition is labeled so that $A_1 \cdot U \neq 0$. Since $A_1$ is dense in $A$, $A_1 \cdot U 
eq 0$. Let $V_1$ be a nonempty open subset of $U \cdot A_1$ such that $0 < \delta(V_1) < e$. Let $a_1 \in V_1 \cdot D(1, f, A_1, B)$. It is easy to prove that there exists an open set $W_1$ in $B$ which contains $f(a_1)$ and such that $f^{-1}(W_1) \cdot A_1 \subset V_1$. Now $f^{-1}(W_1) \cdot A_2$ is an open nonempty subset of $A_2$. Let $V_2$ be a nonempty subset of $f^{-1}(W_1) \cdot A_2$ such that $\delta(V_2) < e$. Let $a_2 \in V_2 \cdot D(1, f, A_2, B)$. Proceeding as before, there exists an open set $W_2$ in $W_1$ such that $W_2 \supset f(a_2)$ and $f^{-1}(W_2) \cdot A_2 \subset V_2$. Note that at this stage $f^{-1}(W_2) \cdot A_i$, $i = 1, 2$, is such that $f[f^{-1}(W_2) \cdot A_i] = W_2$ and $\delta(f^{-1}(W_2)) < e$.

It is easy to see then by induction that we are able to obtain a set $W_k$ in $B$ such that for $Q_i = f^{-1}(W_k) \cdot A_i$, $i = 1, 2, 3, \ldots, k$, it is true that $f(Q_i) = W_k$ and $\delta(Q_i) < e$. It is now easy to verify that if we set $Q = Q_1$, then $Q$ satisfies (i) and (ii) of our preliminary statement (*).

Let $D_e$ be the set of all points $x \in A$ such that $e_k(x) < e$. By (ii) of *, $D_e$ is dense in $A$ for each $e > 0$. Since $e_k$ is upper semi-continuous it follows that the set $D$ of all points $x$ in $A$ such that $e_k(x) = 0$ is also dense in $A$. By 1.3 for each $x \in D$, $f^{-1}(x)$ contains at most $k$ points. Next let $U$ be an open set in $A$. There exists an open set $Q$ in $U$ such that $Q$ satisfies (i) and (ii) of *. Since $D$ as defined above is dense in $A$, there exists an $x \in Q \cdot D$. From the properties of $Q$ and $D$ it follows that $f^{-1}(x)$ consists of exactly $k$ points. Hence $x \in D(k, f, A, B)$ and $D(k, f, A, B)$ is dense in $A$.

The following is easy to verify.

1.7. Let $f$ be an open mapping of a space $A$ onto a space $B$ and suppose $K$ is a compact subset of $A$ such that $f(K) = B$ and $f|_K$ is an irreducible mapping. Let $K^0$ be the interior (relative to $A$) of $K$. Then if $x \in K^0$, $f^{-1}(x)$ consists of exactly one point.

1.8. Let $f$ be an open mapping defined on a compact space $A$ onto a space $B$ such that $D(k, f, A)$ is dense in $A$. Then,

(a) $d(f) = k$ where $d(f) = \sup [N(f(x), f, A)]$ for $x \in A$.
(b) $D(k, f, A)$ is open in $A$.
(c) $f$ is a local homeomorphism at each $x$ in $D(k, f, A)$.
(d) If $F$ is a closed subset of $A$ such that $f(F) = B$ and $f|_F$ is irreducible, then $F^0$, the interior of $F$ (rel. to $A$), is dense in $F$.

(e) Suppose $F$ is a closed subset of $A$ such that $f(F) = B$ and $h$ is a
positive integer such that $1 < h \leq k$, $D(h, f\mid F, F)$ is dense in $F$, and $F^0$ is dense in $F$. Then there exists a decomposition $F = X + Y$ such that $X$ and $Y$ are closed: $X^0$ is dense in $X$; $Y^0$ is dense in $Y$; $X^0 \cdot Y^0 = 0$; $f(X) = f(Y) = B$; $f\mid X$ is irreducible; $D(h - 1, f\mid Y, Y)$ is dense in $Y$.

The proofs of (a), (b), and (c) follow easily from the openness of $f$.

Proof. (d). Let us suppose that $F^0$ is not dense in $F$. Then there exists an open set $U \subset A$ such that $U \cdot F \neq 0$ and $U \cdot F^0 = 0$. Then either $U \cdot F \cdot D(k, f, A)$ is empty or not and we show that either case leads to a contradiction.

Suppose $U \cdot F \cdot D(k, f, A) \neq 0$. Then, by (b), $W = U \cdot D(k, f, A)$ is a nonempty open set which intersects $F$ and since $D(1, f\mid F, F)$ is dense in $F$, there exists a point $y_i \in W \cdot F \cdot D(1, f\mid F, F)$. Let $y_1 + y_2 + y_3 + \cdots + y_k = f(1)(y_i)$ where we note that $y_2 + y_3 + \cdots + y_k \subseteq A - F$. By (c), $f$ is a local homeomorphism at each $y_i$. Hence, there exist open sets $U_i, i = 1, 2, \ldots, k$, such that: $U_i \cdot U_j = 0$ for $i \neq j$, $f(U_i) = f(U_j)$, $f^{-1}f(U_i) = \sum_{i=1}^k U_i$, $f\mid U_i$ is a homeomorphism, $U_i \subset U$, $U_i \subset A - F$ for $i = 2, 3, \ldots, k$. Since $U \cdot F^0 = 0$, there exists a $q \in U_1 \cdot (A - F)$, whence $f^{-1}f(q) \cdot F = 0$ and we have a contradiction since $f(F) = B$. Next, consider the case in which $U \cdot F \cdot D(k, f, A, B) = 0$. There exists a $y \in U \cdot F \cdot D(1, f\mid F, F)$. Since $D(k, f, A)$ is dense in $A$, there exists a sequence $y_i \to y$ such that $y_i \in U \cdot D(k, f, A, B)$. By the hypothesis for this case, none of the $y_i$'s are in $F$. However, since $f(F) = B$ and since $F$ is compact, there exists a sequence $x_n \to z \in F$ such that $x_n \in F$ and $f(x_n) = f(y_n)$. Then since $z \in F$ and $y \in F \cdot D(1, f\mid F, F)$, it follows that $z = y$. Notice that if $y_n \in D(k, f, A, B)$, each point of $f^{-1}f(y_n)$ is also. Hence for some $j$, $x_j \in U \cdot F \cdot D(k, f, A, B)$, a contradiction to the hypothesis for this case.

Proof of (e). There exists a closed subset $X \subset F$ such that $f(X) = B$ and such that $f\mid X$ is an irreducible mapping. Let $Y$ be the closure of $F - X$. We first show that $f(Y) = B$. Suppose $f(Y) \neq B$. Then $f^{-1}(B - f(Y))$ is a nonempty open set such that $f^{-1}(B - f(Y)) \cdot Y = 0$. This leads to a contradiction, for by (d), $X^0$ the interior (rel. to $A$) of $X$ is dense in $X$ and further by hypothesis $D(h, f\mid F, F)$ is dense in $F$. Hence there exists an $x \in f^{-1}(B - f(Y)) \cdot X^0 \cdot D(h, f\mid F, F)$. By 1.7, $f^{-1}f(x) \cdot X = x$. Hence $f^{-1}f(x) \cdot (F - X)$ must contain $h - 1$ points. Thus, $f^{-1}f(x) \cdot Y \neq 0$ and we have a contradiction.

We proceed to prove that $D(h - 1, f\mid Y, Y)$ is dense in $Y$. First note that $Y^0$, the interior (rel. to $A$) of $Y$ is dense in $Y$. Hence, we need show only that $D(h - 1, f\mid Y, Y)$ is dense in $Y^0$. Let $U$ be an open set in $Y^0$. Since $X^0$ is dense in $X$ and $f(X) = B$, then $f(X^0)$ is dense in $B$. Then, since $f$ is open, it is easy to see that $f^{-1}f(X^0)$ is dense in $A$. Hence $U \cdot f^{-1}f(X^0)$ is a nonempty open subset and since
$D(h, f \mid F, F)$ is dense in $F$, it follows that there exists $x \in U \cdot f^{-1}(X^0) \cdot D(h, f \mid F, F)$. By 1.7, $f^{-1}(f(x)) \cdot X$ is a single point, whence since $x \in D(h, f \mid F, F)$, $x \in D(h-1, f \mid Y, Y)$.

1.9. Let $f$ be an open mapping defined on a compact space $A$ onto a space $B$. Let $k$ be a positive integer. Then if $D(k, f, A)$ is dense in $A$, there exists a $k$-fold irreducible decomposition of $A$ relative to $f$.

**Proof.** If $k = 1$, the theorem is trivial. Let $k > 1$. By 1.8 (e), there exists a decomposition $A = A_1 + A_2$ such that $A_1, A_2$ are closed subsets of $A$, $A_1^0 \cdot A_2^0 = 0$, $f(A_1) = f(A_2) = B$, $f \mid A_1$ is irreducible, and $D(k-1, f \mid A_2, A_2)$ is dense in $A_2$. Let $L$ be the collection of all integers $m$ between 2 and $k$ inclusive satisfying the condition that there exists a decomposition $A = \sum_{i=1}^m X_i$ with the following properties:

1. each $X_i$ is closed;  
2. $X_i^0 \cdot X_j^0 = 0$ for $i \neq j$;  
3. $f(X_i) = B$ and $f \mid X_i$ is irreducible for $i=1, 2, 3, \cdots, m-1$;  
4. $f(X_m) = B$ and $f \mid X_m$ satisfies the condition that $D(k-m+1, f \mid X_m)$ is dense in $X_m$;  
5. $X_i^0$ is dense in $X_i$ for $i=1, 2, \cdots, m$. Since $A = A_1 + A_2$ satisfies the above conditions, $2 \in L$. Let $j = \max L$. Now $j = k$, for suppose $j < k$. Then there exists a decomposition $A = Y_1 + Y_2 + \cdots + Y_j$ satisfying conditions (1) through (5). But $f(Y_j) = B$ and $D(k-j+1, f \mid Y_j)$ is dense in $Y_j$ where it is to be noted that $k-j+1 \geq 2$. Hence 1.8 (e) applies to $f \mid Y_j$ and there exists a decomposition $Y_j = Y_j^* + Y_{j+1}$ such that the following conditions are satisfied: $f(Y_j^*) = f(Y_{j+1}) = B$; $f \mid Y_j^*$ is irreducible; $f \mid Y_{j+1}^*$ is such that $D(k-j, f \mid Y_{j+1}^*)$ is dense in $Y_{j+1}^*$; $Y_j^* \cdot Y_{j+1}^* = 0$; $Y_{j+1}^*$ is dense in $Y_{j+1}^*$. But then $A = Y_1 + Y_2 + \cdots + Y_j + Y_{j+1}^*$ satisfies the 5 conditions and $\max L = j+1$, a contradiction.

The following remark is easy to verify.

1.10. Let $f$ be a mapping defined on a compact space $A$ and suppose $f_1$ and $f_2$ are any continuous factors of $f$. Then $f$ is irreducible if and only if $f_1$ and $f_2$ are irreducible mappings.

1.11. Let $f$ be a mapping defined on a compact space $A$ onto a space $B$ and let $f_1$ and $f_2$ be monotone-light factors of $f$. If $D(k, f, A, B)$ is a dense subset of $A$, then $D(1, f_1, A, f_1(A))$ is a dense subset of $A$ and $D(k, f_2, f_1(A), B)$ is a dense subset of $f_1(A)$.

**Proof.** Because of the properties of the monotone-light factors of a mapping, $D(k, f, A, B) \subset D(1, f_1, A, f_1(A))$. Hence $D(1, f_1, A, f_1(A))$ is dense in $A$. Also since $D(k, f, A, B)$ is dense in $A$, $f_1(D(k, f, A, B))$ is dense in $f_1(A)$. But for each $z \in f_1(D(k, f, A, B))$, $N(f_2(z), f_2, f_1(A), B) = k$. Hence $f_1(D(k, f, A, B)) \subset D(k, f_2, f_1(A), B)$ and thus $D(k, f_2, f_1(A), B)$ is also dense in $f_1(A)$.

1.12. Let $f$ be a mapping on a compact space $A$ onto a space $B$ and suppose $f_1, f_2$ are monotone-light open factors of $f$. Then if $D(k, f, A, B)$ is dense in $A$, $A$ has a $k$-fold irreducible decomposition relative to $f$. 

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Proof. Let \( f_1(A) = X \). By 1.11, \( D(1, f_1, A, X) \) is dense in \( A \) and \( D(k, f_2, X, B) \) is dense in \( X \). Hence \( f_1 \) is irreducible and, by 1.9, \( X \) has a \( k \)-fold irreducible decomposition relative to \( f_2 \). Let \( X = \sum_{i=1}^{k} A_i \) be a \( k \)-fold decomposition of \( X \) relative to \( f_2 \). Let \( A_i \) be the closure of \( f_1^{-1}(X_i^0) \) for \( i = 1, 2, \ldots, k \). We show that \( \sum_{i=1}^{k} A_i \) is a \( k \)-fold decomposition of \( A \) relative to \( f \). Since \( f_1(\sum_{i=1}^{k} A_i) = X \) and \( f_1 \) is irreducible, it follows that \( \sum_{i=1}^{k} A_i = A \). From the definition of \( A_i \), \( f_1^{-1}(X_i^0) \) is dense in \( A_i \). Then since \( A_i^0 \cup f_1^{-1}(X_i^0) \) is also dense in \( A_i \). Further, it is easy to see that \( A_i^0 \cdot A_j^0 = 0 \) for \( i \neq j \). Finally, we show that \( f(A_i) = B \) and \( f \mid A_i \) is an irreducible mapping. \( f_1(A_i) \cup f_2 f_1^{-1}(X_i^0) = X_i^0 \). Since \( f_1(A_i) \) is closed and \( X_i^0 \) is dense in \( X_i \), \( f_1(A_i) = X_i \). So \( f_2 f_1(A_i) = f_2(X_i) = B \). Also, \( A_i^0 \cdot D(1, f_1, A^0, X) \subset D(1, f_1^0, A_i, X_i) \). Then since \( A_i^0 \cdot D(1, f_1, A, X) \) is dense in \( A_i \), so also is \( D(1, f_1^0, A_i, X_i) \). Thus \( f_1 \mid A_i \) is irreducible and since \( f_2 \mid X_i \) is also, it follows that \( f \mid A_i \) is as well.

As a corollary to the above proof we have the following

**Corollary.** If \( f \) is a mapping defined on a compact space \( A \) with monotone light open factorization \( f = f_2 f_1 \) and if \( f_1 \) is irreducible and the space \( f_1(A) \) possesses a \( k \)-fold irreducible decomposition relative to \( f_2 \), then \( A \) possesses a \( k \)-fold irreducible decomposition relative to \( f \).

By using 10.4 and 10.41 of [4] and 1.12 and 1.6 we obtain the

1.13. **Theorem.** Let \( f \) be a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum. Then the set of points \( D \) at which \( f \) is exactly \( k \)-to-one is dense in \( A \) if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \).

**Corollary.** Under the same hypothesis as the theorem, if for each \( x \) in \( A \), \( f^{-1}(f(x)) \) consists of at least \( k \) points, then \( f \) is \( k \)-fold irreducible if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \).

2. **Examples and applications.**

2.1. Let \( f \) be an irreducible mapping defined on a compact space \( A \) which admits a monotone light-open factorization \( f = f_2 f_1 \). Then \( f \) is monotone.

**Proof.** By 1.10, \( f_2 \) is irreducible and hence by Corollary 3 of 1.4, \( f_2 \) is a homeomorphism. Hence \( f \) is monotone.

In preparation for the next result we prove the following lemma.

2.2. Let \( f \) be a mapping of a simple closed curve \( A \) onto a simple closed curve \( B \). Then \( f \) is monotone if and only if the set \( D \) of points \( x \) in \( B \) for which \( f^{-1}(x) \) is a single point is dense in \( B \).
Proof. If $f$ is monotone it is easy to see that $D$ is dense in $B$. Conversely, suppose $D$ is dense in $B$ and let $p \in B$. Since $D$ is dense in $B$, we can find sequences $p_i \to p$ and $p_i^* \to p$ such that $p$ is between $p_i$ and $p_i^*$ for each $i$ and such that all the $p_i$'s and $p_i^*$'s are in $D$. We may suppose that $\text{arc} (p_i p_i^*) \subseteq \text{arc} (p_{i+1} p_{i+1}^*)$ for each $i$. Further it is easy to prove that for each $i$, $f^{-1}(\text{arc} (p_i p_i^*))$ is an arc. Now since $p = \prod_{i=1}^{\infty} \text{arc} (p_i p_i^*)$ and since $\prod_{i=1}^{\infty} f^{-1}(\text{arc} (p_i p_i^*))$ is connected it follows that $f^{-1}(p) = \bigcap_{i=1}^{\infty} f^{-1}(\text{arc} (p_i p_i^*)) = \prod_{i=1}^{\infty} f^{-1}(\text{arc} (p_i p_i^*))$ is connected and hence $f$ is monotone.

2.3. Let $f$ be a quasi-monotone mapping defined on a boundary curve $A$ such that for each $x$ in $A$, $f^{-1}(x)$ consists of at least $k$ points. Then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one.

Proof. We prove the necessity. Let $f_1$ and $f_2$ be monotone, light open factors of $f$. If $f$ is $k$-fold irreducible, then $f_1$ is an irreducible monotone mapping and hence since $A$ is a boundary curve, it is clear that $f_1$ must be one-to-one and hence a homeomorphism. Then $f_2$ is $k$-fold irreducible and hence since $f_2$ is open $f_2$ must be $k$-to-one. The converse is obvious.

The next example follows easily from 1.13 and a similar argument to that used in 2.3.

2.4. Let $f$ be a quasi-monotone mapping defined on a boundary curve $A$. Then if $A$ has a $k$-fold irreducible decomposition relative to $f$, $f$ is a light-open mapping.

The next result follows from 2.4 and a theorem of G. T. Whyburn for open mappings defined on a simple closed curve. See X, 1.2 in [3].

2.5. Let $f$ be a quasi-monotone mapping of $A$ onto $B$ where $A$ and $B$ are simple closed curves. Then $f$ is $k$-fold irreducible (or equivalently in this case, $A$ has a $k$-fold irreducible decomposition relative to $f$) if and only if $f$ is topologically equivalent to the transformation $w = z^k$ defined on the circle $|z| = 1$.

The next result is a consequence of X, 6.3 in [3] and 1.8 (a) and 1.13.

2.6. Let $f$ be an open mapping defined on a compact 2-manifold $A$. Then $A$ has a $k$-fold irreducible decomposition relative to $f$ if and only if the degree (as defined in [3]) is $k$.

References

G. T. Whyburn