1. Introduction. Let $f$ be a mapping of a space $A$ onto a space $B$. Let $N(y, f, A, B)$, $y \in B$, be the number of points in $f^{-1}(y)$. Let $k$ be a positive integer. According to G. T. Whyburn [2], $f$ is said to be $k$-fold irreducible provided that for $y \in B$, $N(y, f, A, B) \geq k$, and if $F$ is a closed proper subset of $A$, then for some $z \in B$, $N(z, f|F, F, B) < k$. It should be noted that if $f$ is 1-fold irreducible, then $f$ is strongly irreducible in accordance with the terminology of [3] or irreducible in accordance with [2].

In this paper there is introduced the notion of a $k$-fold irreducible decomposition of a space relative to a mapping as follows.

**Definition.** Let $f$ be a mapping defined on a compact space $A$ onto a space $B$. $A$ is said to possess a $k$-fold irreducible decomposition relative to $f$ provided that there exists a decomposition $A = \sum_{i=1}^{k} A_i$ where each $A_i$ is a nonempty closed subset of $A$ and the decomposition satisfies the following conditions:

(i) $A_i^0$, the interior (rel. to $A$) of $A_i$, is dense in $A_i$.
(ii) $A_i \cdot A_j^0 = 0$, $i \neq j$.
(iii) $f(A_i) = B$ and $f|A_i$ is an irreducible mapping.

We further define $D(k, f, A, B)$ to be the set of all $x \in A$ for which $N(f(x), f, A, B) = k$. When there is no chance of confusion we shall write $D(k, f, A, B)$ as $D(k, f, A)$.

In 1.1 through 1.5 some definitions and results of G. T. Whyburn which will be needed are listed. In the remaining part of §1, we obtain results concerning $k$-fold irreducible decomposition of a space relative to certain types of mappings. One of the principal results (1.13) states that if $f$ is a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum, then the set of points $D$ at which $f$ is exactly $k$-to-one is dense in $A$ if and only if $A$ has a $k$-fold irreducible decomposition relative to $f$. A corollary of this theorem is that for the same hypothesis as the theorem, if for each $x \in A$, $f^{-1}(f(x))$ consists of at least $k$ points, then $f$ is $k$-fold irreducible if and only if $A$ has a $k$-fold irreducible decomposition relative to $f$.
In §2, some examples of $k$-fold irreducible decomposition are given. It is shown that if $f$ is a quasi-monotone mapping defined on a boundary curve $A$ such that for each $x \in A$, $f^{-1}f(x)$ consists of at least $k$ points, then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one. Also, if $f$ is a quasi-monotone mapping defined on a boundary curve which has a $k$-fold irreducible decomposition relative to $f$, then $f$ is a light open mapping. Finally, it is shown that for an open-mapping $f$ defined on a compact 2-manifold $A$, $A$ has a $k$-fold irreducible decomposition relative to $f$ if and only if the degree of $f$, as defined in [3], is $k$.

For a general reference to topological terms and results used in this paper see [3]. All spaces are assumed to be metric.

1.1. For any set $A$ in a metric space and any positive $k$, define:
$$e_k(A) = \inf \left\{ \max \delta(A_i) \right\}$$
for all decompositions $A = \sum_{i=1}^{k} A_i$ of $A$ into $k$ nonempty subsets $A_i$. Here $\delta$ stands for diameter.

For $f$, a mapping defined on a metric space $X$, define
$$e_k(x) = e_k(f^{-1}f(x)), \quad x \in X.$$  

1.2. If the mapping $f$ defined on $X$ generates an upper-semi-continuous decomposition of $X$, the function $e_k(x)$ is upper-semi-continuous.

Hence the theorem applies if $X$ is compact.

1.3. $e_k(x) = 0$ if and only if $N(f(x), f, A) \leq k$.

1.4. Let $f$ be a mapping defined on a compact space $A$. Suppose for each $x \in A$, $N(f(x), f, A) \geq k$. Then $f$ is $k$-fold irreducible if and only if the set of points $x \in A$ such that $N(f(x), f, A) = k$ is dense in $A$ (i.e. $D(k, f, A)$).

Corollary 1. $f$ is irreducible if and only if the set $D$ of all points $x \in A$ with $x = f^{-1}f(x)$ is dense in $A$.

Corollary 2. If $f$ is open, then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one.

Corollary 3. If $f$ is open, then $f$ is irreducible if and only if $f$ is a homeomorphism.

1.5. Let $f$ be a mapping defined on a compact space $A$ onto a space $B$. Then there exists a compact set $A' \subset A$ such that $f(A') = B$ and $f|A'$ is an irreducible mapping.

1.6. Theorem. Let $f$ be a mapping defined on a compact space $A$ onto a space $B$. If $f$ possesses a $k$-fold irreducible decomposition relative to $f$, then $D(k, f, A, B)$ is dense in $A$.

Proof. We first prove the following.
(*) Let $U$ be an open set in $A$ and let $e > 0$. Then there exists an open set $Q$ in $U$ such that (i) if $q \in Q$, then $f^{-1}(q)$ contains at least $k$ points and (ii) for each $q \in Q$, $e_k(q) < e$, where $e_k(q)$ is as defined in 1.1.

Toward the end of proving this statement, let $A = \sum_{i=1}^k A_i$ be a $k$-fold irreducible decomposition relative to $f$ and we suppose further that this decomposition is labeled so that $A_1$. $U \neq 0$. Since $A_1^0$ is dense in $A_1$, $A_1^0 \cdot U \neq 0$. Let $V_1$ be a nonempty open subset of $U \cdot A_1^0$ such that $0 < \delta(V_1) < e$. Let $a_1 \in V_1 \cdot D(1, f, A_1, B)$. It is easy to prove that there exists an open set $W_1$ in $B$ which contains $f(a_1)$ and such that $f^{-1}(W_1) \cdot A_1 \subset V_1$. Now $f^{-1}(W_1) \cdot A_2^0$ is an open nonempty subset of $A_2$. Let $V_2$ be a nonempty subset of $f^{-1}(W_1) \cdot A_2^0$ such that $\delta(V_2) < e$. Let $a_2 \in V_2 \cdot D(1, f, A_2, B)$. Proceeding as before, there exists an open set $W_2$ in $W_1$ such that $W_2 \supset f(a_2)$ and $f^{-1}(W_2) \cdot A_2 \subset V_2$. Note that at this stage $f^{-1}(W_2) \cdot A_i$, $i = 1, 2$, is such that $f[f^{-1}(W_2) \cdot A_i] = W_2$ and $\delta(f^{-1}(W_2) \cdot A_i) < e$.

It is easy to see then by induction that we are able to obtain a set $W_k$ in $B$ such that for $Q_i = f^{-1}(W_k) \cdot A_i$, $i = 1, 2, 3, \ldots, k$, it is true that $f(Q_i) = W_k$ and $\delta(Q_i) < e$. It is now easy to verify that if we set $Q = Q_1$, then $Q$ satisfies (i) and (ii) of our preliminary statement (*).

Let $D_e$ be the set of all points $x \in A$ such that $e_k(x) < e$. By (ii) of *, $D_e$ is dense in $A$ for each $e > 0$. Since $e_k$ is upper semi-continuous it follows that the set $D$ of all points $x$ in $A$ such that $e_k(x) = 0$ is also dense in $A$. By 1.3 for each $x \in D$, $f^{-1}(x)$ contains at most $k$ points. Next let $U$ be an open set in $A$. There exists an open set $Q$ in $U$ such that $Q$ satisfies (i) and (ii) of *. Since $D$ as defined above is dense in $A$, there exists an $x \in Q \cdot D$. From the properties of $Q$ and $D$ it follows that $f^{-1}(x)$ consists of exactly $k$ points. Hence $x \in D(k, f, A, B)$ and $D(k, f, A, B)$ is dense in $A$.

The following is easy to verify.

1.7. Let $f$ be an open mapping of a space $A$ onto a space $B$ and suppose $K$ is a compact subset of $A$ such that $f(K) = B$ and $f|K$ is an irreducible mapping. Let $K^0$ be the interior (relative to $A$) of $K$. Then if $x \in K^0$, $f^{-1}(x) \cdot K$ consists of exactly one point.

1.8. Let $f$ be an open mapping defined on a compact space $A$ onto a space $B$ such that $D(k, f, A)$ is dense in $A$. Then,

(a) $d(f) = k$ where $d(f) = \sup [d(f(x), f, A)]$ for $x \in A$.
(b) $D(k, f, A)$ is open in $A$.
(c) $f$ is a local homeomorphism at each $x$ in $D(k, f, A)$.
(d) If $F$ is a closed subset of $A$ such that $f(F) = B$ and $f|F$ is irreducible, then $F^0$, the interior of $F$ (rel. to $A$), is dense in $F$.
(e) Suppose $E$ is a closed subset of $A$ such that $f(E) = B$ and $h$ is a
positive integer such that \(1 < h \leq k\), \(D(h, f \mid F, F)\) is dense in \(F\), and \(F^0\) is dense in \(F\). Then there exists a decomposition \(F = X + Y\) such that \(X\) and \(Y\) are closed: \(X^0\) is dense in \(X\); \(Y^0\) is dense in \(Y\); \(X^0 \cdot Y^0 = 0\); \(f(X) = f(Y) = B\); \(f \mid X\) is irreducible; \(D(h - 1, f \mid Y, Y)\) is dense in \(Y\).

The proofs of (a), (b), and (c) follow easily from the openness of \(f\).

Proof. (d). Let us suppose that \(F^0\) is not dense in \(F\). Then there exists an open set \(U \subset A\) such that \(U \cdot F \neq 0\) and \(U \cdot F^0 = 0\). Then either \(U \cdot F \cdot D(k, f, A)\) is empty or not and we show that either case leads to a contradiction.

Suppose \(U \cdot F \cdot D(k, f, A) \neq 0\). Then, by (b), \(W = U \cdot D(k, f, A)\) is a nonempty open set which intersects \(F\) and since \(D(1, f \mid F, F)\) is dense in \(F\), there exists a point \(y_1 \in W \cdot F \cdot D(1, f \mid F, F)\). Let \(y_1 + y_2 + y_3 + \cdots + y_k = f^{-1}(y_1)\) where we note that \(y_2 + y_3 + \cdots + y_k \subset A - F\).

By (c), \(f\) is a local homeomorphism at each \(y_i\). Hence, there exist open sets \(U_i, i = 1, 2, \ldots, k\), such that: \(U_i \cdot U_j = 0\) for \(i \neq j\), \(f(U_i) = f(U_j)\), \(f^{-1}(U_i) = \sum_{i=1}^{k} U_i\), \(f \mid U_i\) is a homeomorphism, \(U_i \subset U\), \(U_i \subset A - F\) for \(i = 2, 3, \ldots, k\). Since \(U \cdot F = 0\), there exists a \(q \in U_1 \cdot (A - F)\), whence \(f^{-1}(q) \cdot F = 0\) and we have a contradiction since \(f(F) = B\).

Next, consider the case in which \(U \cdot F \cdot D(k, f, A, B) = 0\). There exists a \(y \in U \cdot F \cdot D(1, f \mid F, F)\). Since \(D(k, f, A)\) is dense in \(A\), there exists a sequence \(y_i \rightarrow y\) such that \(y_i \in U \cdot D(k, f, A, B)\). By the hypothesis for this case, none of the \(y_i\)'s are in \(F\). However, since \(f(F) = B\) and since \(F\) is compact, there exists a sequence \(x_n \rightarrow z \in F\) such that \(x_n \in F\) and \(f(x_n) = f(y_n)\). Then since \(z \in F\) and \(y \in F \cdot D(1, f \mid F, F)\), it follows that \(z = y\). Notice that if \(y_n \in D(k, f, A, B)\), each point of \(f^{-1}(y_n)\) is also. Hence for some \(j, x_j \in U \cdot F \cdot D(k, f, A, B)\), a contradiction to the hypothesis for this case.

Proof of (e). There exists a closed subset \(X \subset F\) such that \(f(X) = B\) and such that \(f \mid X\) is an irreducible mapping. Let \(Y\) be the closure of \(F - X\). We first show that \(f(Y) = B\). Suppose \(f(Y) \neq B\). Then \(f^{-1}(B - f(Y))\) is a nonempty open set such that \(f^{-1}(B - f(Y)) \cdot Y = 0\). This leads to a contradiction, for by (d), \(X^0\) the interior (rel. to \(A\)) of \(X\) is dense in \(X\) and further by hypothesis \(D(h, f \mid F, F)\) is dense in \(F\). Hence there exists an \(x \in f^{-1}(B - f(Y)) \cdot X^0 \cdot D(h, f \mid F, F)\). By 1.7, \(f^{-1}(x) \cdot X = x\). Hence \(f^{-1}(x) \cdot (F - X)\) must contain \(h - 1\) points. Thus, \(f^{-1}(x) \cdot Y \neq 0\) and we have a contradiction.

We proceed to prove that \(D(h - 1, f \mid Y, Y)\) is dense in \(Y\). First note that \(Y^0\), the interior (rel. to \(A\)) of \(Y\) is dense in \(Y\). Hence, we need show only that \(D(h - 1, f \mid Y, Y)\) is dense in \(Y^0\). Let \(U\) be an open set in \(Y^0\). Since \(X^0\) is dense in \(X\) and \(f(X) = B\), then \(f(X^0)\) is dense in \(B\). Then, since \(f\) is open, it is easy to see that \(f^{-1}(f(X^0))\) is dense in \(A\). Hence \(U \cdot f^{-1}(f(X^0))\) is a nonempty open subset and since
\(D(h, f | F, F)\) is dense in \(F\), it follows that there exists \(x \in U \cdot f^{-1}(X^0) \cdot \overline{D(h, f | F, F)}\). By 1.7, \(f^{-1}\cdot f(x) \cdot X\) is a single point, whence since \(x \in \overline{D(h, f | F, F)}, x \in \overline{D(h-1, f | Y, Y)}\).

1.9. Let \(f\) be an open mapping defined on a compact space \(A\) onto a space \(B\). Let \(k\) be a positive integer. Then if \(D(k, f, A)\) is dense in \(A\), there exists a \(k\)-fold irreducible decomposition of \(A\) relative to \(f\).

Proof. If \(k = 1\), the theorem is trivial. Let \(k > 1\). By 1.8 (e), there exists a decomposition \(A = A_1 + A_2\) such that \(A_1, A_2\) are closed subsets of \(A\), \(A_1^0 \cdot A_2^0 = 0\), \(f(A_1) = f(A_2) = B\), \(f|A_1\) is irreducible, and \(D(k-1, f|A_2, A_2)\) is dense in \(A_2\). Let \(L\) be the collection of all integers \(m\) between 2 and \(k\) inclusive satisfying the condition that there exists a decomposition \(A = \sum_{i=1}^{m} X_i\) with the following properties: (1) each \(X_i\) is closed; (2) \(X_i^0 \cdot X_j^0 = 0\) for \(i \neq j\); (3) \(f(X_i) = B\) and \(f|X_i\) is irreducible for \(i = 1, 2, 3, \ldots, m-1\); (4) \(f(X_m) = B\) and \(f|X_m\) satisfies the condition that \(D(k-m+1, f|X_m)\) is dense in \(X_m\); (5) \(X_i^0\) is dense in \(X_i\) for \(i = 1, 2, \ldots, m\). Since \(A = A_1 + A_2\) satisfies the above conditions, \(2 \in L\). Let \(j = \max L\). Now \(j < k\), for suppose \(j < k\). Then there exists a decomposition \(A = Y_1 + Y_2 + \cdots + Y_j\) satisfying conditions (1) through (5). But \(f(Y_j) = B\) and \(D(k-j+1, f|Y_j)\) is dense in \(Y_j\) where it is to be noted that \(k-j+1 \geq 2\). Hence 1.8 (e) applies to \(f|Y_j\) and there exists a decomposition \(Y_j = Y_j^* + Y_j^{*+1}\) such that the following conditions are satisfied: \(f(Y_j^*) = f(Y_j^{*+1}) = B\); \(f|Y_j^*\) is irreducible; \(f|Y_j^{*+1}\) is such that \(D(k-j, f|Y_j^{*+1})\) is dense in \(Y_j^{*+1}\); \(Y_j^* \cdot Y_j^{*+1} = 0\); \(Y_j^{*+1}\) is dense in \(Y_j^{*+1}\). But then \(A = Y_1 + Y_2 + \cdots + Y_j + Y_j^{*+1}\) satisfies the 5 conditions and \(\max L \geq j+1\), a contradiction.

The following remark is easy to verify.

1.10. Let \(f\) be a mapping defined on a compact space \(A\) and suppose \(f_1\) and \(f_2\) are any continuous factors of \(f\). Then \(f\) is irreducible if and only if \(f_1\) and \(f_2\) are irreducible mappings.

1.11. Let \(f\) be a mapping defined on a compact space \(A\) onto a space \(B\) and let \(f_1\) and \(f_2\) be monotone-light factors of \(f\). If \(D(k, f, A, B)\) is a dense subset of \(A\), then \(D(1, f_1, A, f_1(A))\) is a dense subset of \(A\) and \(D(k, f_2, f_1(A), B)\) is a dense subset of \(f_1(A)\).

Proof. Because of the properties of the monotone-light factors of a mapping, \(D(k, f, A, B) \subseteq D(1, f_1, A, f_1(A))\). Hence \(D(1, f_1, A, f_1(A))\) is dense in \(A\). Also since \(D(k, f, A, B)\) is dense in \(A\), \(f_1(D(k, f, A, B))\) is dense in \(f_1(A)\). But for each \(z \in f_1(D(k, f, A, B))\), \(N(f_2(z), f_2, f_1(A), B) = k\). Hence \(f_1(D(k, f, A, B)) \subseteq D(k, f_2, f_1(A), B)\) and thus \(D(k, f_2, f_1(A), B)\) is also dense in \(f_1(A)\).

1.12. Let \(f\) be a mapping on a compact space \(A\) onto a space \(B\) and suppose \(f_1, f_2\) are monotone-light open factors of \(f\). Then if \(D(k, f, A, B)\) is dense in \(A\), \(A\) has a \(k\)-fold irreducible decomposition relative to \(f\).
Proof. Let \( f_1(A) = X \). By 1.11, \( D(1, f_1, A, X) \) is dense in \( A \) and \( D(k, f_2, X, B) \) is dense in \( X \). Hence \( f_1 \) is irreducible and, by 1.9, \( X \) has a \( k \)-fold irreducible decomposition relative to \( f_2 \). Let \( X = \sum_{i=1}^{k} X_i \) be a \( k \)-fold decomposition of \( X \) relative to \( f_2 \). Let \( A_i \), be the closure of \( f_1^{-1}(X_i) \) for \( i = 1, 2, \ldots, k \). We show that \( \sum_{i=1}^{k} A_i \) is a \( k \)-fold decomposition of \( A \) relative to \( f_1 \). Since \( f_1(\sum_{i=1}^{k} A_i) = X \) and \( f_1 \) is irreducible, it follows that \( \sum_{i=1}^{k} A_i = A \). From the definition of \( A_i \), \( f_1^{-1}(X_i) \) is dense in \( A_i \). Then since \( A_i \supseteq f_1^{-1}(X_i) \), \( A_i \) is also dense in \( A_i \). Further, it is easy to see that \( A_i \cdot A_j = 0 \) for \( i \neq j \). Finally, we show that \( f(A_i) = B \) and \( f \mid A_i \) is an irreducible mapping. \( f_1(A_i) \supseteq f_1 f_1^{-1}(X_i) = X_i \). Since \( f_1(A_i) \) is closed and \( X_i \) is dense in \( X_i \), \( f_1(A_i) = X_i \). So \( f f_1(A_i) = f_2(X_i) = B \). Also, \( A_i \cdot D(1, f_1, A, X) \subseteq D(1, f_1 \mid A_i, A_i, X_i) \). Then since \( A_i \cdot D(1, f_1, A, X) \) is dense in \( A_i \), so also is \( D(1, f_1 \mid A_i, A_i, X_i) \). Thus \( f_1 \mid A_i \) is irreducible and since \( f_2 \mid X_i \) is also, it follows that \( f \mid A_i \) is as well.

As a corollary to the above proof we have the following

**Corollary.** If \( f \) is a mapping defined on a compact space \( A \) with monotone light open factorization \( f = f_2 f_1 \) and if \( f_1 \) is irreducible and the space \( f_1(A) \) possesses a \( k \)-fold irreducible decomposition relative to \( f_2 \), then \( A \) possesses a \( k \)-fold irreducible decomposition relative to \( f \).

By using 10.4 and 10.41 of [4] and 1.12 and 1.6 we obtain the

1.13. **Theorem.** Let \( f \) be a quasi-interior mapping defined on a compact space or a quasi-monotone mapping defined on a locally connected continuum. Then the set of points \( D \) at which \( f \) is exactly \( k \)-to-one is dense in \( A \) if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \).

**Corollary.** Under the same hypothesis as the theorem, if for each \( x \) in \( A \), \( f^{-1}(x) \) consists of at least \( k \) points, then \( f \) is \( k \)-fold irreducible if and only if \( A \) has a \( k \)-fold irreducible decomposition relative to \( f \).

**2. Examples and applications.**

2.1. Let \( f \) be an irreducible mapping defined on a compact space \( A \) which admits a monotone light-open factorization \( f = f_2 f_1 \). Then \( f \) is monotone.

**Proof.** By 1.10, \( f_2 \) is irreducible and hence by Corollary 3 of 1.4, \( f_2 \) is a homeomorphism. Hence \( f \) is monotone.

In preparation for the next result we prove the following lemma.

2.2. Let \( f \) be a mapping of a simple closed curve \( A \) onto a simple closed curve \( B \). Then \( f \) is monotone if and only if the set \( D \) of points \( x \) in \( B \) for which \( f^{-1}(x) \) is a single point is dense in \( B \).
Proof. If $f$ is monotone it is easy to see that $D$ is dense in $B$. Conversely, suppose $D$ is dense in $B$ and let $p \in B$. Since $D$ is dense in $B$, we can find sequences $p_i \to p$ and $p_i^* \to p$ such that $p$ is between $p_i$ and $p_i^*$ for each $i$ and such that all the $p_i$'s and $p_i^*$'s are in $D$. We may suppose that $\text{arc } (p_i p_i^*) \subseteq \text{arc } (p_{i+1} p_{i+1}^*)$ for each $i$. Further it is easy to prove that for each $i$, $f^{-1}(\text{arc } (p_i p_i^*))$ is an arc. Now since $p = \prod_{i=1}^{\infty} \text{arc } (p_i p_i^*)$ and since $\prod_{i=1}^{\infty} f^{-1}(\text{arc } (p_i p_i^*))$ is connected it follows that $f^{-1}(p) = f^{-1}(\prod_{i=1}^{\infty} \text{arc } (p_i p_i^*)) = \prod_{i=1}^{\infty} f^{-1}(\text{arc } (p_i p_i^*))$ is connected and hence $f$ is monotone.

2.3. Let $f$ be a quasi-monotone mapping defined on a boundary curve $A$ such that for each $x$ in $A$, $f^{-1}(x)$ consists of at least $k$ points. Then $f$ is $k$-fold irreducible if and only if $f$ is $k$-to-one.

Proof. We prove the necessity. Let $f_1$ and $f_2$ be monotone, light open factors of $f$. If $f$ is $k$-fold irreducible, then $f_1$ is an irreducible monotone mapping and hence since $A$ is a boundary curve, it is clear that $f_1$ must be one-to-one and hence a homeomorphism. Then $f_2$ is $k$-fold irreducible and hence since $f_2$ is open $f_2$ must be $k$-to-one. The converse is obvious.

The next example follows easily from 1.13 and a similar argument to that used in 2.3.

2.4. Let $f$ be a quasi-monotone mapping defined on a boundary curve $A$. Then if $A$ has a $k$-fold irreducible decomposition relative to $f$, $f$ is a light-open mapping.

The next result follows from 2.4 and a theorem of G. T. Whyburn for open mappings defined on a simple closed curve. See X, 1.2 in [3].

2.5. Let $f$ be a quasi-monotone mapping of $A$ onto $B$ where $A$ and $B$ are simple closed curves. Then $f$ is $k$-fold irreducible (or equivalently in this case, $A$ has a $k$-fold irreducible decomposition relative to $f$) if and only if $f$ is topologically equivalent to the transformation $w = z^k$ defined on the circle $|z| = 1$.

The next result is a consequence of X, 6.3 in [3] and 1.8 (a) and 1.13.

2.6. Let $f$ be an open mapping defined on a compact 2-manifold $A$. Then $A$ has a $k$-fold irreducible decomposition relative to $f$ if and only if the degree (as defined in [3]) is $k$.

References

G. T. Whyburn