A THEOREM ABOUT TOPOLOGICAL $n$-CELLS

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1. Introduction. Let $\mathbb{R}^n$ be Euclidean $n$-space. We define $I^n$ to be the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $0 \leq x_j \leq 1$ for $j = 1, \ldots, n$. Any set which is homeomorphic to $I^n$ will be called a topological $n$-cell. The result obtained in this paper is the following:

Theorem. If $G$ is a nonempty, open, connected subset of $\mathbb{R}^n$, then there exists a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \cdots$ of topological $n$-cells such that $\bigcup_{m=1}^{\infty} E_m = G$.

We point out that our topological $n$-cells are closed sets. It is obvious that in general $G$ could not be represented as the union of a nondecreasing sequence of “open topological $n$-cells.” We also point out that we do not place any type of simple connectedness restriction on $G$.

The sets $E_m$ which we construct in the proof of our theorem are not only topological $n$-cells, but each is the union of a finite number of $n$-dimensional cubes. The same thing is true for the cells $E(i, j)$.

2. Proof of the theorem. If $k$ is a non-negative integer and $m_1, \ldots, m_n$ are integers, we define $\sigma(k; m_1, \ldots, m_n)$ to be the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ for which $m_j 2^{-k} \leq x_j \leq (m_j + 1)2^{-k}$ for $j = 1, \ldots, n$. We define $\Sigma$ to be the set of all $n$-cells $\sigma(k; m_1, \ldots, m_n)$ which are contained in $G$. $\Sigma$ is countable and we arrange its members in a sequence $S_1, S_2, \ldots$.

We are going to define topological $n$-cells $E(i, j)$ for all positive integers $i, j$ in such a way that:

1. $E(i, j) \subset E(i, j+1)$ and $E(i, j) \subset E(i+1, j)$ for all $i, j$;
2. $P_i = \bigcup_{j=1}^{\infty} E(i, j)$ can be expressed as the union of a finite number of members of $\Sigma$;
3. $S_i \subset P_i$ for each $i$.

Once we have defined sets $E(i, j)$ having the above properties, it is easy to see that $E(1, 1) \subset E(2, 2) \subset E(3, 3) \subset \cdots$ and that $\bigcup_{m=1}^{\infty} E(m, m) = G$. Thus we can define $E_m = E(m, m)$ and obtain topological $n$-cells which have the desired properties.

We think of the sets $E(i, j)$ as being the elements of an infinite matrix, with $i$ being the “row variable” and $j$ being the “column variable.” The sets $E(i, j)$ are to be defined a row at a time.

We need the following lemma.

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Lemma 1. Suppose that a row \( E(m, 1), E(m, 2), \ldots \) has been defined in such a way that (1) and (2) are satisfied, and suppose that \( J \) is an \((n-1)\)-cell on the boundary of \( P_m \). Then there exists an integer \( k \) and an \((n-1)\)-cell \( J^* \) such that \( J^* \subset J \) and \( J^* \subset E(m, j) \) for all \( j \geq k \).

Proof. Let \( C(j) = J \cap E(m, j) \) for each positive integer \( j \). Then each \( C(j) \) is a closed subset of \( J \). Since \( J = \bigcup_{j=1}^{\infty} C(j) \) and \( J \) is of the second category with respect to itself, there exists an integer \( k \) such that \( C(k) \) contains an open set relative to \( J \). We may choose \( J^* \) to be any \((n-1)\)-cell which is contained in this open set.

We begin the definitions of the sets \( E(i, j) \) by defining \( E(1, j) = S_1 \) for each positive integer \( j \).

Now assume that \( m > 1 \) and that for all \( i < m \) we have defined the topological \( n \)-cells \( E(i, j) \) for all \( j \), and that these sets have been defined in such a way that (1), (2), and (3) are satisfied. We wish to define next sets \( E(m, 1), E(m, 2), \ldots \). There are five cases which we consider.

Case 1. \( S_m \cap P_{m-1} = \emptyset \).

Let \( C \) be the component of \( G - P_{m-1} \) that contains \( S_m \), and let \( J \) be an \((n-1)\)-cell which is contained in \( C \cap P_{m-1} \). We use the lemma to obtain an integer \( k \) and an \((n-1)\)-cell \( J^* \subset J \) such that \( J^* \subset E(m-1, j) \) for all \( j \geq k \). It is easy to see that we can unite a finite number of members of \( \Sigma \) to form a topological \( n \)-cell \( F \) such that: \( F \subset C \), \( F \cap P_{m-1} \) is an \((n-1)\)-cell which is contained in \( J^* \), and \( F \cap S_m \) is an \((n-1)\)-cell.

It follows that \( E(m-1, j) \cup F \cup S_m \) is a topological \( n \)-cell for \( j \geq k \).

We define \( E(m, j) = E(m-1, j) \) for \( j < k \) and \( E(m, j) = E(m-1, j) \cup F \cup S_m(2^{-q} \delta) \) for \( j \geq k \).

Case 2. \( P_{m-1} \) does not contain interior points of \( S_m \), and \( P_{m-1} \cap S_m \) contains some \((n-1)\)-cell \( J \).

We again use the lemma to obtain an integer \( k \) and an \((n-1)\)-cell \( J^* \) such that \( J^* \subset J \cap E(m-1, j) \) for all \( j \geq k \).

If \( \epsilon \) is a positive number of the form \( 2^{-q} \), \( q \) a positive integer, we define \( S_m(\epsilon) \) to be the closure of the set obtained by subtracting from \( S_m \) all members of \( \Sigma \) which have edges of length \( \epsilon \) and which contain points of \( S_m \cap P_{m-1} \). It is obvious that for sufficiently small \( \epsilon \), \( S_m(\epsilon) \) is a topological \( n \)-cell. We assume that \( \delta = 2^{-q} \) is so small that \( S_m(\epsilon) \) is a topological \( n \)-cell for all \( \epsilon < \delta \).

It is possible to unite a finite number of members of \( \Sigma \) to form a topological \( n \)-cell \( F \) for which: \( F \subset S_m \), \( F \cup S_m(\epsilon) \) is a topological \( n \)-cell for \( \epsilon < \delta \), \( F \cap P_{m-1} \) is a topological \((n-1)\)-cell which is contained in \( J^* \). It follows that \( E(m-1, j) \cup F \cup S_m(2^{-q} \delta) \) is a topological \( n \)-cell for \( j \geq k \).

We define \( E(m, j) = E(m-1, j) \cup F \cup S_m(2^{-q} \delta) \) for \( j \geq k \).
Case 3. $P_{m-1}$ does not contain interior points of $S_m$, and $S_m \cap P_{m-1}$ contains only cells of dimension less than $n-1$.

We choose an $(n-1)$-cell $J$ contained in the intersection of $P_{m-1}$ with the boundary of the component $C$ of $G - P_{m-1}$ which contains the interior of $S_m$. We also require that $J$ be disjoint from $S_m \cap P_{m-1}$. An integer $k$ and an $(n-1)$-cell $J^*$ are chosen as in case 2, and sets $S_m(\epsilon)$ are defined as in case 2. We also choose $\delta$ as in case 2. It is possible to unite a finite number of members of $\Sigma$ so as to form a topological $n$-cell $F$ such that $F \cap S_m(\epsilon)$ is a topological $n$-cell for all $\epsilon < \delta$ and such that $F \cap P_{m-1}$ is an $(n-1)$-cell which is contained in $J^*$. We define the sets $E(m, j)$ as in case 2.

Case 4. $P_{m-1}$ contains interior points of $S_m$, but does not contain all of $S_m$.

It is easy to see that the closure of $S_m - P_{m-1}$ can be expressed as the union of a finite set $\sigma_1, \ldots, \sigma_p$ of elements of $\Sigma$. We start with the sequence $E(m-1, 1), E(m-1, 2), \ldots$ and apply the construction outlined in case 2 $p$ times to construct successively $p$ sequences of topological $n$-cells. This may be done in such a way that the union of the elements of the $j$th sequence constructed will contain $j$ members of the set $\sigma_1, \ldots, \sigma_p$. We take the last of the $p$ sequences which we construct to be the sequence $E(m, 1), E(m, 2), \ldots$. It is easily seen that $S_m \subseteq \bigcup_{j=1}^{p} E(m, j)$.

Case 5. $S_m \subseteq P_{m-1}$.

We define $E(m, j) = E(m-1, j)$ for all $j$.

In the following corollary, $\Pi_j(G - H)$ is the $j$th homotopy group of $G - H$.

**Corollary 1.** If $G$ is a nonempty, open, connected subset of $R^n$, then there exists a subset $H$ of $G$ such that the dimension of $H$ is less than $n$, $G - H$ is open and connected, and $\Pi_j(G - H) = 0$ for all $j$.

**Proof.** Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be topological $n$-cells such that $\bigcup_{m=1}^{\infty} E_m = G$. We let $B_m$ be the boundary of $E_m$ and let $U_m = E_m - B_m$. We define $H = \bigcup_{m=1}^{\infty} \bigcap_{j=m}^{\infty} B_j$.

Since each $B_j$ is closed and of dimension $n-1$, $\bigcap_{j=m}^{\infty} B_j$ is closed and of dimension less than $n$ for each $m$. It follows from the theorem on countable unions of closed sets that $H$ is also of dimension less than $n$.

It is easy to verify that $G - H = \bigcup_{m=1}^{\infty} U_m$. Since each $U_m$ is open and connected and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$, $G - H$ is also open and connected.

Let $S$ be a $j$-sphere, and let $f$ be a continuous function on $S$ into $G - H$. Then $f(S)$ is compact, and it follows that $f(S) \subseteq U_m$ for some $m$. Since $U_m$ is homeomorphic to an open $n$-cell in $R^n$, $f$ is homotopic
to a constant in $U_m$, and hence also in $G-H$. Thus $\Pi_f(G-H) = 0$.

For the special case of open connected subsets of $\mathbb{R}^n$, $n \leq 3$, we can strengthen Corollary 1 and show that the set $H$ can be chosen so that $G-H$ is an open topological 3-cell. Since the result is relatively trivial for $n=2$ and is completely trivial for $n=1$, we restrict our attention to the case $n=3$. We use a theorem of Moise to prove:

**Lemma 2.** If $I'$ and $I''$ are closed 3-cells, $J'$ and $J''$ are closed polyhedral 3-cells, $I'$ is a subset of the interior of $I''$, $J'$ is a subset of the interior of $J''$, and $h$ is a piecewise linear homeomorphism of $I'$ onto $J'$; then, $h$ can be extended to a piecewise linear homeomorphism which maps $I''$ onto $J''$.

**Proof.** There exists a piecewise linear homeomorphism $k$ of $I''$ onto $J''$. We define $A = k^{-1}h(I')$. It follows from a theorem of Moise (see [1, Theorem 1]) that there exists a piecewise linear homeomorphism $\phi$ of $I''$ onto $I''$ which takes $A$ onto $I'$. It is easy to see that $\phi k^{-1}h$ maps $I'$ onto $I'$, and can thus be extended in an obvious manner to a piecewise linear homeomorphism $\psi$ of $I''$ onto $I''$. We define $h^* = k\phi^{-1}\psi$. Then $h^*$ is a piecewise linear homeomorphism which is an extension of $h$ and which maps $I''$ onto $J''$.

**Corollary 2.** If $G$ is an open, connected subset of $\mathbb{R}^3$, then there exists a subset $H$ of $G$ of dimension less than 3 for which there exists a piecewise linear homeomorphism of $\mathbb{R}^3$ onto $G-H$.

**Proof.** It follows from our theorem (and its proof) that there exist closed polyhedral 3-cells $E_1 \subseteq E_2 \subseteq \cdots$ such that $\bigcup_{m=1}^{\infty} E_m = G$. We define $U_m$, $B_m$, and $H$ as in Corollary 1. It is easily seen that there exist closed polyhedral 3-cells $J_m \subseteq U_m$ such that $\bigcup_{m=1}^{\infty} J_m = \bigcup_{m=1}^{\infty} U_m = G-H$ and such that $J_m$ is a subset of the interior of $J_{m+1}$ for each $m$.

We define $I_m$ to be the set of all points $(x_1, x_2, x_3)$ in $\mathbb{R}^3$ for which $\max(|x_1|, |x_2|, |x_3|) \leq m$. There exists a piecewise linear homeomorphism $h_1$ of $I_1$ onto $J_1$. If $h_m$ has been defined so as to be a piecewise linear homeomorphism of $I_m$ onto $J_m$, then we use Lemma 2 to extend $h_m$ to a piecewise linear homeomorphism $h_{m+1}$ on $I_{m+1}$ onto $J_{m+1}$. If $x \in \mathbb{R}^3$, then there exists $m$ such that $x \in I_m$ and we define $h(x) = h_m(x)$. The function $h$ is a piecewise linear homeomorphism of $\mathbb{R}^3$ onto $G-H$.

**Bibliography**