ON THE METRIZABILITY OF THE BUNDLE SPACE

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It has been shown by Yu. M. Smirnov (see [1, p. 13, Theorem 3]) that a Hausdorff space $X$ is metrizable if and only if $X$ is paracompact and has an open cover each of whose members is metrizable.

Using this, we prove: If \( \{X, B, \pi, Y, \mathcal{U}, \phi, G\} \) is a fibre bundle (see [2, p. 7]) whose base space $B$ and fibre $Y$ are metrizable, then the bundle space $X$ is also metrizable.

First, if $B$ and $Y$ are metrizable, then $X$ is Hausdorff and has an open cover each of whose members is metrizable, namely \( \{\pi^{-1}(U) \mid U \in \mathcal{U}\} \), since if $U \in \mathcal{U}$ then $U \times Y$, and hence $\pi^{-1}(U)$, is metrizable.

Second, $X$ is paracompact, for let $\mathcal{C}$ be any open cover for $X$. Let $\mathcal{G}$ be a locally finite refinement of $\mathcal{U}$, and let $\mathcal{W}$ be a closure refinement of $\mathcal{G}$ which is also locally finite; define $\lambda: \mathcal{W} \to \mathcal{G}$ a function such that, for each $W$, if $W \in \mathcal{W}$ then $W \subseteq \lambda(W)$. Let for each $V \in \mathcal{G}$

\[ \mathcal{D}_V = \{ C \cap \pi^{-1}(V) \mid C \in \mathcal{C} \} \]

and let $\mathcal{D}_V^*$ be a locally finite refinement of $\mathcal{D}_V$ (this exists, since $\pi^{-1}(V)$ is metric, hence paracompact, for each $V \in \mathcal{G}$). For $W \in \mathcal{W}$ define

\[ \mathcal{E}_W = \{ D \cap \pi^{-1}(W) \mid D \in \mathcal{D}_V^* \} \]

and let $\mathcal{F} = \bigcup_{W \in \mathcal{W}} \mathcal{E}_W$. Clearly $\mathcal{F}$ refines $\mathcal{C}$, and it is thus sufficient to show that $\mathcal{F}$ is locally finite.

Let $x \in X$; then since $\mathcal{W}$ is locally finite, so is $\{\pi^{-1}(W) \mid W \in \mathcal{W}\}$; therefore there is a neighborhood $A$ of $x$ which meets only a finite number of members of $\{\pi^{-1}(W) \mid W \in \mathcal{W}\}$, say $\pi^{-1}(W_1), \ldots, \pi^{-1}(W_n)$. For $1 \leq i \leq n$ define $B_i$ a neighborhood of $x$ as follows:

1. If $x$ is a member of $\pi^{-1}(W_i)$ let $B_i$ be a neighborhood of $x$ which meets only a finite number of members of $\mathcal{E}_{W_i}$;

2. If $x$ is a member of the boundary of $\pi^{-1}(W_i)$ let $B_i$ be a neighborhood of $x$ which meets only a finite number of members of $\mathcal{D}_{\lambda(W_i)}^*$, hence only a finite number of members of $\mathcal{E}_{W_i}$;

3. If $x$ is not in the closure of $\pi^{-1}(W_i)$ let $B_i$ be a neighborhood of $x$ which meets only a finite number of members of $\mathcal{E}_{W_i}$;

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x which does not meet $\pi^{-1}(W_i)$, hence which meets no members of $\mathcal{E}_{W_i}$.

Then $A \cap \bigcap_{i=1}^{n} U_i$ is a neighborhood of $x$ which meets only a finite number of members of $\mathcal{J}$, since it meets only members of $\mathcal{E}_{W_1}, \ldots, \mathcal{E}_{W_n}$, and at most a finite number of each of these.

**References**


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