ON THE INFINITESIMAL GENERATOR OF A SEMIGROUP OF
POSITIVE TRANSFORMATIONS WITH LOCAL
CHARACTER CONDITION

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The study of one-dimensional diffusion processes leads to the consideration of the following structure.

Let \( \{ T_t \} (t > 0) \) be a one-parameter semigroup \([2]\) of linear operators on the space \( C \) of bounded continuous functions on a closed interval \( I \) satisfying the conditions:

(a) If \( f(x) \geq 0 \) for all \( x \) in \( I \), then \( T_t f(x) \geq 0 \) for all \( x \) and all \( t \) (positivity).

(b) If \( f \) vanishes throughout a neighborhood of \( x \), then \( T_t f(x) = o(t) \) (local character).

(c) \( T_t 1 = 1 \) for all \( t \) (conservation of mass).

Let \( \Omega \) be the infinitesimal generator of \( \{ T_t \} \). That is,

\[
\Omega f(x) = \lim_{t \to 0^+} \frac{T_t f(x) - f(x)}{t}
\]

defined on the set \( D \) of those \( f \) in \( C \) for which the limit exists uniformly in \( x \). We shall also consider the expression (1) wherever the limit exists pointwise.

In this paper the form of \( \Omega \) at a point \( x \) is determined, but the problem of characterizing \( \Omega \) in the large is left open.

W. Feller \([1]\) has shown that if the functions in \( D \) are twice differentiable, then, with the exception of certain singular points, \( \Omega \) corresponds to a differential operator of the form

\[
\Omega f(x) = a(x) \frac{d^2 f(x)}{dx^2} + b(x) \frac{df(x)}{dx}
\]

where \( a(x) \geq 0 \). However, Feller was dissatisfied with the arbitrary injection of differentiability conditions since the conditions (a), (b), and (c) are independent of the metric on \( I \). He conjectured that, with only the conditions (a), (b), and (c), \( \Omega \) must be essentially a second order differential operator.

Consider a fixed point \( x \) in \( I \). We shall consider only those functions \( f \) for which \( f(x) = 0 \). This is no real restriction since (c) implies...
$\Omega f = \Omega (f - \lambda)$ wherever $\Omega f$ exists. The expression “near $x$” will be used to mean “in some neighborhood of $x$.” The notation $f = o(h)$ will be used whenever, for arbitrary $\epsilon > 0$, $|f(z)| \leq \epsilon |h(z)|$ for all $z$ near $x$. The functions $f_-$ and $f_+$, defined by

$$f_-(z) = \begin{cases} f(z) & \text{for } z < x \\ 0 & \text{for } z \geq x \end{cases} \quad \text{and} \quad f_+ = f - f_-,$$

will also be useful.

We begin with two fundamental lemmas due to Feller [1].

**Lemma A.** If $f \geq 0$ near $x$ and $\Omega f(x)$ exists, then $\Omega f(x) \geq 0$. Thus, $\Omega f(z) \geq 0$ if $f$ has a local minimum at $z$ and $\Omega f(z) \leq 0$ if $f$ has a local maximum at $z$.

**Proof.** $|f| - f = 0$ near $x$. Hence $T_t |f| (x) - T_t f(x) = o(t)$ by (b). Since $T_t |f| \geq 0$ by (a), and $\Omega f(x) = \lim_{t \to 0} T_t f(x)/t = \lim_{t \to 0} T_t |f| (x)/t$, $\Omega f(x) \geq 0$.

**Lemma B.** If $\Omega v(x)$ exists, $v \geq 0$ near $x$, and $f = o(v)$, then $\Omega f(x)$ exists and $\Omega f(x) = 0$.

**Proof.** For arbitrary $\epsilon > 0$, $-\epsilon v \leq f \leq \epsilon v$ near $x$. From (b), $-\epsilon T_t v(x) \leq T_t f(x) + o(t) \leq \epsilon T_t v(x)$. Dividing by $t$ and letting $t$ approach 0,

$$- \epsilon \Omega v(x) \leq \liminf_{t \to 0} \frac{T_t f(x)}{t} \leq \limsup_{t \to 0} \frac{T_t f(x)}{t} \leq \epsilon \Omega v(x).$$

Since $\epsilon$ may be arbitrarily small, the conclusion holds.

In order to introduce a metric we prove

**Theorem 1.** If $f$ is in $D$ and $\Omega f(x) \neq 0$, then $f$ is strictly monotone in a right (left) neighborhood of $x$.

**Proof.** Suppose $f$ is not strictly monotone in any right neighborhood of $x$. Then, by induction, we can choose a double sequence of points $\{y_n, z_n\}$ such that $y_n \downarrow x$, $z_n \downarrow x$, $f(y_n) = f(z_n)$, and $x \leq y_{n+1} < z_{n+1} < y_n < z_n$ for all $n$. Thus, there exist points $r_n$ and $s_n$ in the interior of $(y_{n+1}, z_n)$ such that $f$ has a local minimum at $r_n$ and a local maximum at $s_n$. By Lemma A, we have $\Omega f(r_n) \geq 0$ and $\Omega f(s_n) \leq 0$. Since $r_n \downarrow x$, $s_n \downarrow x$, and since $\Omega f$ is continuous for $f$ in $D$, it follows that $\Omega f(x) = 0$, a contradiction. A symmetric argument gives the theorem for a left neighborhood.

An immediate result of Theorem 1 is

**Corollary 1.** If $f$ is in $D$ and the zeros of $f$ are dense at $x$, then $\Omega f(x) = 0$. 

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If there is a function $g$ in $D$ with $\Omega g(x) \neq 0$, then every $f$ in $D$ has right and left derivatives (not necessarily finite) at $x$ with respect to $g$. The existence of unilateral derivatives is given by

**Theorem 2.** If $f$ and $g$ are in $D$ and $\Omega g(x) \neq 0$, then

$$\lim_{\varepsilon \to +} \inf - \frac{f(z)}{g(z)} = \lim_{\varepsilon \to +} \sup - \frac{f(z)}{g(z)}$$

and

$$\lim_{\varepsilon \to +} \inf - \frac{f(z)}{g(z)} = \lim_{\varepsilon \to +} \sup - \frac{f(z)}{g(z)} .$$

**Proof.** Let $P$ denote the set of real numbers $\rho$ for which $\rho g(z) \leq f(z)$ near $x$, and $Q$ denote the set for which $f(z) \leq \rho g(z)$ near $x$. Then $\rho^- = \sup P \rho$ and $\rho^+ = \inf Q \rho$ are the lower and upper right derivatives of $f$ with respect to $g$. Suppose $\rho^- \neq \rho^+$. Then consider any finite $\rho$ such that $\rho^- < \rho < \rho^+$ and $\rho \neq \Omega f(x)/\Omega g(x)$. Such a $\rho$ belongs to neither $P$ nor $Q$. Thus $f(z) - \rho g(z) < 0$ at a set of points dense at $x$ and $f(z) - \rho g(z) > 0$ at a dense set. Hence, the zeros of $f - \rho g$ are dense at $x$. By Corollary 1, $\Omega f(x) = \rho \Omega g(x)$, a contradiction. Therefore, $\rho^- = \rho^+$. A similar proof holds for the left derivatives.

We now seek to express $\Omega$ in terms of such unilateral derivatives.

**Theorem 3.** Let $f$ and $g$ belong to $D$. If either $f_+ = o(g)$ and $f_+ \geq 0$ near $x$ or $f_- = o(g)$ and $f_- \geq 0$ near $x$, then $\Omega f(x) \geq 0$.

**Proof.** If $g$ has its zeros dense at $x$ from the right, then $f$ does likewise for $f_+ = o(g)$, so $\Omega f(x) = 0$ by Corollary 1. Otherwise we may assume that $g > 0$ in a deleted right neighborhood of $x$. Then for each $\varepsilon > 0$, $0 \leq f_+ \leq \varepsilon g_+$ near $x$. For $z$ in an arbitrary deleted right neighborhood of $x$ we may set $\varepsilon = f(z)/g(z)$. Then $f_+ - \varepsilon g_+ \leq 0$ near $x$ and $f - \varepsilon g(x) = f - \varepsilon g(z) = 0$, so $f - \varepsilon g$ has a local minimum at some point $y$, $x < y < z$. By Lemma A, $\Omega (f - \varepsilon g)(y) \geq 0$, so $\Omega f(y) \geq \varepsilon \Omega g(y)$. As $z$ converges to $x$, $\varepsilon$ converges to 0 and $y$ converges to $x$ giving $\Omega f(x) \geq 0$. A similar proof gives the theorem with $f_+$ replaced by $f_-$. We can now extend Lemma B to give

**Theorem 4.** Suppose either $g \geq 0$ or $fg \geq 0$ near $x$ for $f$ and $g$ in $D$. Then $f = o(g)$ implies $\Omega f(x) = 0$.

**Proof.** Suppose $\Omega f(x) \neq 0$. Then we cannot have $g \geq 0$ near $x$, for this would contradict Lemma B. Hence, $fg \geq 0$ near $x$ with $g$ having neither a maximum nor minimum at $x$. By Theorem 1, $f$ is strictly monotone near $x$. Thus, either $f_- \leq 0$ and $f_+ \geq 0$ or $f_- \geq 0$ and $f_+ \leq 0$.
near \( x \). By Theorem 3, \( \Omega f(x) = 0 \), contradicting our initial assumption.

By means of the preceding theorem, Theorem 2 can be sharpened to give

**Theorem 5.** If \( u \) is in \( D \), \( \Omega u(x) \neq 0 \), and \( u \) is monotone near \( x \), then every \( f \) in \( D \) has finite unilateral derivatives with respect to \( u \).

**Proof.** Suppose \( f \) has an infinite right derivative, \( \sigma_+ = \pm \infty \). Then \( f \) must have a finite left derivative, \( \sigma_- \). Otherwise, \( u = o(f) \) and Theorem 4 would give \( \Omega u(x) = 0 \), a contradiction.

Choose \( \sigma \) so that \( \sigma_- < \sigma < \infty \) if \( \sigma_+ = \infty \) and \( -\infty < \sigma < \sigma_- \) if \( \sigma_+ = -\infty \). Let \( v = f - \sigma u \). Then, since \( u_+ = o(v) \), Lemma B gives \( \Omega u_+(x) = 0 \). So \( \Omega u(x) = \Omega u_-(x) \). But \( \Omega u_-(x) \leq 0 \) if \( u \) is increasing, by Lemma A. So \( \Omega u(x) < 0 \). However, \( u_+ \geq 0 \) near \( x \) and \( u_+ = o(f) \) implying \( \Omega u(x) > 0 \) by Theorem 3, a contradiction.

A similar proof holds for the left derivative.

We can now express \( \Omega f \) at \( x \) in terms of an increasing function \( u \) and a function \( v \) having a minimum at \( x \).

**Theorem 6.** At \( x \) one of the following four cases must hold:

(I) \( \Omega f(x) = 0 \) for all \( f \) in \( D \).

(II) For all \( u \) in \( D \) monotone near \( x \), \( \Omega u(x) = 0 \). There exists \( v \) in \( D \) with \( v \geq 0 \) near \( x \) and \( \Omega v(x) > 0 \). Then, for each \( f \) in \( D \) there exists a number \( \rho \) such that

\[
(2) \quad f = \rho v + o(v)
\]

and

\[
(3) \quad \Omega f(x) = \rho \Omega v(x).
\]

(III) For all \( v \) in \( D \) with \( v \geq 0 \) near \( x \), \( \Omega v(x) = 0 \). There exists \( u \) in \( D \) with \( u \) strictly increasing near \( x \) and \( \Omega u(x) \neq 0 \). Then, for each \( f \) in \( D \) there exists a number \( \sigma \) such that

\[
(4) \quad f = \sigma u + o(u)
\]

and

\[
(5) \quad f(x) = \sigma \Omega u(x).
\]

(IV) There exist \( u \) and \( v \) in \( D \) with \( u \) strictly increasing near \( x \), \( \Omega u(x) \neq 0 \), \( v \geq 0 \) near \( x \), and \( \Omega v(x) > 0 \) such that each \( f \) in \( D \) can be expressed in the form

\[
(6) \quad f = \sigma u + \rho v + h
\]

where \( h = o(u) = o(v) \). Then
\[ \Omega f(x) = \sigma \Omega u(x) + \rho \Omega v(x). \]

**Proof.** If case (I) does not hold, there exist functions \( g \) in \( D \) with \( \Omega g(x) \neq 0 \). Such functions are either strictly monotone near \( x \) or have a maximum (or minimum) at \( x \), by Theorem 1. If all such \( g \) are of the latter type, we have case (II); if the former type, we have case (III).

In case (II) consider any \( f \) in \( D \). If \( f \) had no derivative with respect to \( v \), we could choose a finite \( \rho \) such that (i) \( \rho^- < \rho < \rho^+ \) and (ii) \( \rho \neq \Omega f(x)/\Omega v(x) \). By (ii), \( \Omega (f - \rho v)(x) \neq 0 \). So \( f - \rho v \) is unilaterally monotone near \( x \), by Theorem 1. By Theorem 2, \( f - \rho v \) has unilateral derivatives with respect to \( v \) and these derivatives, \( \rho^- - \rho \) and \( \rho^+ - \rho \), are of opposite sign, by (i). Thus, since \( v \geq 0 \) near \( x \), \( f - \rho v \) is monotone near \( x \). From the assumptions of case (II), \( \Omega (f - \rho v)(x) = 0 \) which contradicts (ii). Hence, (2) holds. (3) follows from Lemma B.

In case (III) each \( f \) has a finite derivative \( \sigma \) with respect to \( u \). Otherwise, there would exist a finite \( \sigma \) such that (i) \( \sigma^- < \sigma < \sigma^+ \) and (ii) \( \sigma \neq \Omega f(x)/\Omega u(x) \). Since \( f \) has finite unilateral derivatives \( \sigma_-, \sigma_+ \) with respect to \( u \) by Theorem 5, (i) implies either \( f - \sigma u \geq 0 \) or \( f - \sigma u \leq 0 \) near \( x \). Hence, (III) gives \( \Omega (f - \sigma u)(x) = 0 \) which contradicts (ii). Thus, we obtain (4) with \( \sigma = \sigma_- = \sigma_+ \). If \( f - \sigma u \) is not monotone near \( x \), then \( \Omega (f - \sigma u)(x) = 0 \) by (III). If \( f - \sigma u \) is monotone near \( x \), then \( \Omega (f - \sigma u)(x) = 0 \) by Theorem 4. Hence, (5) holds.

If the unilaterally monotone functions \( g \) for which \( \Omega g(x) \neq 0 \) are of two types, some monotone near \( x \) and others with extreme values at \( x \), then we may choose a function of each type. In particular, choose \( u \) such that \( u \) is increasing near \( x \) and \( \Omega u(x) \neq 0 \), and choose \( v \) such that \( v \geq 0 \) near \( x \) and \( \Omega v(x) > 0 \). We then have (IV), which we treat in two cases: one in which \( v \) has a derivative with respect to \( u \) and the other in which \( v \) has no derivative with respect to \( u \).

Case (IV-1). \( v = o(u) \). In this case every \( f \) in \( D \) has a derivative \( \sigma \) with respect to \( u \). For, suppose \( \sigma_-, \sigma_+ \). Then \( s = f - 2^{-1}(\sigma_- + \sigma_+)u \) has unilateral derivatives \( 2^{-1}(\sigma_- - \sigma_+) \) and \( 2^{-1}(\sigma_+ - \sigma_-) \) which differ in sign. So \( s \) has a maximum (or minimum) at \( x \) and \( \Omega v(x) > 0 \). Lemma B gives \( \Omega v(x) = 0 \), a contradiction.

Let \( s = f - \sigma u \), so \( s = o(u) \). If \( s \) had unequal derivatives \( \rho^- < \rho^+ \) with respect to \( v \), we could choose \( \rho \) such that \( \rho^- < \rho < \rho^+ \) and \( \rho \neq \Omega s(x)/\Omega v(x) \). Thus, \( \Omega (s - \rho v)(x) \neq 0 \). So \( s - \rho v \) is unilaterally monotone near \( x \), by Theorem 1. By Theorem 2, \( s - \rho v \) has unilateral derivatives with respect to \( v \) and these derivatives, \( \rho^- - \rho \) and \( \rho^+ - \rho \), are of opposite sign. Since \( v \geq 0 \) near \( x \), \( s - \rho v \) is monotone near \( x \). But \( s - \rho v = o(u) \), so \( \Omega (s - \rho v)(x) = 0 \), by Theorem 4, a contradiction. Hence, let \( \rho = \rho^- = \rho^+ \) which must be finite. For, if \( \rho = \infty \) (or \( \rho = -\infty \), then
s ≥ 0 (or s ≤ 0) near x, since v ≥ 0 near x, and v = o(s). Theorem 4 would give Ωv(x) = 0, a contradiction. We thus have (6) for case (IV-1).

Case (IV-2). v has no derivative with respect to u (β_- < β_+). We may assume β_+ = 1 and β_- = −1. For, given any w in D with w ≥ 0 near x, Ωw(x) > 0, and unilateral derivatives (α_- < α_+) with respect to u, we can define

\[ v = \frac{2}{\alpha_+ - \alpha_-} \left\{ w - \frac{1}{2} (\alpha_+ + \alpha_-)u \right\}. \]

Thus, v has β_+ = 1, β_- = −1 and v ≥ 0 near x. Choose β such that −1 < β < 1 with sign such that βΩu(x) > 0. Then, v − βu ≥ 0 near x, so Ω(v − βu)(x) ≥ 0 by Lemma A. Thus Ωv(x) > 0.

Now, for f in D with unilateral derivatives σ_−, σ_+ with respect to u, let \( \sigma = 2^{-1}(\sigma_+ + \sigma_-) \) and \( \rho = 2^{-1}(\sigma_+ - \sigma_-) \). Then it is easily verified by taking right and left derivatives that \( f - \sigma u - \rho v = o(u) = o(v) \), thus giving (6). (7) follows from Lemma B.

Thus, Ω will correspond to a second order differential operator at x whenever it is possible to choose u and v such that u^2 = v + o(v).

REFERENCES

1. W. Feller, On positivity preserving semigroups of transformations on C[r_1, r_2], Annales de la Société Polonaise de Mathématique (1952).

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