Corollary. If \( k = 1 \) and \( *e_0x^2e = x \), then \( *axe_0b = *ae_0xb \) for any \( a \) and \( b \).

Proof. 
\[
*axe_0b = *a*e_0e,e_{r-1}xe_0b \quad \text{(by (2) with } m = 1) \\
= *ae_0*e_{r-1}xe_0b \\
= *ae_0xb \quad \text{(by Theorem Q with } m = 1) .
\]

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TWO THEOREMS ON FINITELY GENERATED GROUPS

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Let \( G \) be a group generated by a finite subgroup \( H \) and an element \( b \) of finite order. If \( H \) commutes elementwise with \( b \) (for this we shall write \([h, b] = e\) for every \( h \in H \) where \([h, b]\) designates \( hbh^{-1}b^{-1} \)), then clearly \( G \) is finite and \( b \) is in the center of \( G \).

We consider here the case where, for every \( h \in H \), \([h, b]b = e\), and prove the following theorem:

Theorem. Let \( G \) be generated by the finite subgroup \( H \) and the element \( b \) of finite order and, for every \( h \in H \), let \([h, b]b = e\). Then \( G \) is finite and \( b \) is in the nil radical of \( G \).

Proof. For \( i = 1, 2, \ldots, n \) let \( h_i \) be the elements of \( H \). Then \( h_i^{-1}bh_i \) are all the conjugates of \( b \); for \( bh^{-1}bh^{-1} = h^{-1}bh \) by virtue of the hypothesis \([h, b]b = e\).

It follows from the fact that a finite set of conjugates generate a finite normal subgroup (cf. [1]) that \( b \) is contained in a finite normal subgroup \( K \) of \( G \). But \( H \) is finite and hence so also is \( G/K \); and then finally \( G \) is finite.

Furthermore since \( b \) is in the center of \( K \), \( b \) is in the nil radical of \( G \) as was asserted.

We can deduce another result from the fact that \([g, b]b = e\) for every \( g \in G \) implies that \( b \) is in the center of a normal subgroup of \( G \).

Theorem. Let \( G \) be a finitely generated group with the property that if \( b_1, \ldots, b_n \) are the generators of \( G \), then \([g, b_i]b_i = e\) for every \( g \in G \) and for \( i = 1, 2, \ldots, n \). Then \( G \) is nilpotent of class at most \( n \). If furthermore the \( b_i \) are of finite order then \( G \) is finite.

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Proof. For \(i=1, 2, \cdots, n\) let \(B_i\) be the normal subgroup of \(G\) in which \(b_i\) is central. Since the \(B_i\) are normal subgroups of \(G\), so is each of the intersections \(B_{i_1} \cap \cdots \cap B_{i_r}\) normal in \(G\). For \(j=1, \cdots, n\) let \(A_j\) represent the subgroup of \(G\) generated by the product of all possible intersections of \(j\) of the \(B_i\) at a time; i.e., \(A_1 = B_1 B_2 \cdots B_n\), \(A_2 = (B_1 \cap B_2)(B_1 \cap B_3) \cdots (B_{n-1} \cap B_n)\), etc., and \(A_n = B_1 \cap B_2 \cap \cdots \cap B_n\).

Then \(A_n\) is in the center of \(G\); for \(A_n\) commutes elementwise with all the generators of \(G\). And for each \(r=1, \cdots, n\), \(A_{r-1}/A_r\) is in the center of \(G/A_r\). For each component \(B_{i_1} \cap \cdots \cap B_{i_{r-1}}\) commutes elementwise with \(b_{i_1}, \cdots, b_{i_{r-1}}\) and \((B_{i_1} \cap \cdots \cap B_{i_{r-1}}) \cap B_i \subset A_r\); hence modulo \(A_r\) each component of \(A_{r-1}\) is in the center of \(G/A_r\) and consequently \(A_{r-1}/A_r\) is in the center of \(G/A_r\) as asserted. Hence \(G\) is nilpotent of class at most \(n\). The finiteness of \(G\) follows immediately from this if the \(b_i\) are of finite order.

Corollary. If \(G\) is a finitely generated group all of whose elements have order 3, then \(G\) is finite (cf. [2]).

For let \(a\) and \(b\) be any two elements of \(G\). Then \([\partial b, a]a = bab\partial aba\^2b\^2a\^2 = (bab)(bab)(b\^2a\^2)(b\^2a\^2)(b\^2a\^2) = e\).

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