A NOTE ON RINGS WITH CENTRAL NILPOTENT ELEMENTS

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The theorem proved in this note, and its corollary, are designed to improve, and in a sense to bring to a more satisfactory completion, a theorem which we proved in [1].

We prove the

**Theorem.** Let $R$ be a ring such that for every element $x$ in $R$ there exists an integer $n = n(x)$, and a polynomial $p(t) = p_n(t)$ with integer coefficients which depend on $x$, such that $x^{n+1}p(x) = x^n$. If further all the nilpotent elements of $R$ are in the center of $R$, then $R$ is commutative.

**Proof.** Since $x^{n+1}p(x) = x^n$, we have that $(x^2p(x) - x)x^{n-1} = 0$ (we can assume that $n > 1$ for this could always be achieved by multiplying both sides of the equation by $x$). Now, each term of $(x^2p(x) - x)^{n-1}$ involves $x$ to a power which is at least $n - 1$; therefore $(x^2p(x) - x)^n = (x^2p(x) - x)(x^2p(x) - x)^{n-1} = 0$. Since $x^2p(x) - x$ is nilpotent, by assumption it must lie in the center of $R$. This is true for every $x$ in $R$, so it follows from [2] that $R$ is commutative.

**Corollary.** Let $R$ be a ring such that every element of $R$ generates a finite subring. If the nilpotent elements of $R$ are all in the center, then $R$ is commutative.

**Proof.** Since $x$ in $R$ generates a finite subring, $x^n = x^m$ for some $n > m$, so the corollary is immediate from the theorem. This corollary is a direct generalization of the second theorem in [1].

**References**


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