SOME REMARKS ON \(\nu\)-TRANSITIVE RINGS
AND LINEAR COMPACTNESS

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Johnson in [3] has introduced the concept of a \(\nu\)-transitive ring which generalizes the notion of a dense ring of linear transformations. We give necessary and sufficient conditions that an abstract ring be isomorphic to a \(\nu\)-transitive ring which contains finite-valued linear transformations. The condition (2) used here is a modification of one used by Baer [1] in his characterization of the endomorphism ring of a primary Abelian operator group. This condition is also related to the linear compactness of a ring considered as a right module over itself. This enables one to conclude that a primitive ring with minimal ideals which is linearly compact (in any topology in which it is a topological ring) is the ring of all linear transformations of a vector space, and that a primitive Banach algebra is linearly compact only when it is finite dimensional.

A ring \(E(F, A)\) of linear transformations of the vector space \(A\) over the division ring \(F\) is called \(\nu\)-transitive if to every set of less than \(\aleph\) elements \(a_j\) of \(A\), linearly independent over \(F\), and any set of elements \(b_j\) of \(A\), in one-one correspondence with the \(a_j\), there exists a transformation \(\sigma\) in \(E(F, A)\), such that \(a_j\sigma = b_j\) for all \(j\).

Let \(K\) be an abstract ring and \(P\) an arbitrary subset thereof. The right ideal of all elements \(k\) in \(K\) which satisfies \(Pk = 0\) shall be called a right annulet. Now let \(W = W(K)\) be the class of all right annulets which are cross-cuts of a finite number of maximal right annulets of \(K\). By a \(W\)-coset is meant a coset of an ideal in the set \(W(K)\). If \(E(F, A)\) is any ring of linear transformations and \(S\) is a subspace of \(A\), we denote by \(R(S)\) the totality of transformations \(\sigma \in E(F, A)\) satisfying \(S\sigma = 0\).

Theorem. A ring \(K\) is isomorphic to a \(\nu\)-transitive ring containing linear transformations of finite rank if and only if:

1. The socle of \(K\) is not a zero ring and is contained in every non-zero two-sided ideal of \(K\).

2. If \(Q\) is any set of \(W\)-cosets with the finite intersection property, then any subset of \(Q\) containing less than \(\aleph\) elements has a nonvacuous intersection.

Proof. Condition (1) is necessary and sufficient that \(K\) be isomorphic to a dense ring of linear transformations \(E(F, A)\), contain-
ing linear transformations of finite rank (cf. e.g. [4, Theorem 6.1]). Now assume (2), and let \( \phi \) be an arbitrary linear transformation of \((F, A)\) and \( S \) an arbitrary subspace of \( A \) of rank \( < \aleph_\alpha \). Let \( \{a_i\} \) denote a basis for \( S \). Since \( E \) is dense, there exist \( \phi_i \in E \) such that \( \phi_i = \phi \) on the one-dimensional subspace \( F_{a_i} \). An ideal \( J \in W(E) \) if and only if \( J = R(T) \) where \( T \) is a finite-dimensional subspace of \( A \) [2, p. 19]. Thus the cosets \( \{R(F_{a_i}) + \phi_i\} \) are \( W \)-cosets and possess the finite-intersection property by density of \( E \). Hence by (2), there exists \( \sigma \in E \) such that \( \sigma \in R(F_{a_i}) + \phi_i \) for all \( i \). That is, \( a_i(\sigma - \phi_i) = 0 \), \( a_i\sigma = a_i\phi \), which is \( \nu \)-transitivity.

Assume now that \( E(F, A) \) is \( \nu \)-transitive. Let \( \{R(S_i) + \sigma_i\} \) be \( W \)-cosets with the finite intersection property. It may be assumed without loss of generality that the cardinal number of this set of cosets is less than \( \aleph_\alpha \). Then the subspace \( S = \bigcup S_i \) has rank less than \( \aleph_\alpha \) since each \( S_i \) is finite-dimensional. By Zorn’s Lemma, any set of generators of \( S \) contains a basis of \( S \), and we can therefore find a basis \( \{u_i\} \) of \( S \) such that each \( u_i \) is contained in at least one \( S_i \). For each \( i \), select one \( S_i \) containing \( u_i \), and find by \( \nu \)-transitivity \( \sigma \in E \) such that \( u_i \sigma = u_i \sigma_i \) for each \( i \) (where \( \sigma_i \) is the linear transformation associated with the \( S_i \) containing \( u_i \)). If \( u_i \) is also contained in \( S_j \), then \( u_i \sigma = u_i \sigma_j \) since the finite intersection property of \( \{R(S_i) + \sigma_i\} \) implies \( \sigma_i = \sigma_j \) on \( S_i \cap S_j \). We must prove \( \sigma = \sigma_{i_0} \) on \( S_{i_0} \) for any \( i_0 \). Let \( x \in S_{i_0} \) and write\(^1\)

\[ x = \alpha_1 u_{i_1} + \alpha_2 u_{i_2} + \cdots + \alpha_n u_{i_n}, \quad u_{i_j} \in S_{i_j}. \]

Then \( x\sigma = \sum \alpha_j(u_{i_j}\sigma_{i_j}) \). By the finite intersection property, we can select \( \tau \) in \( \bigcap_{i=0}^n (R(S_{i_j}) + \sigma_{i_j}) \). Since \( \tau = \sigma_{i_0} \) on \( S_{i_0} \), we have \( x\tau = x\sigma_{i_0} \).

But \( x\tau = \sum \alpha_j(u_{i_j}\tau) = \sum \alpha_j(u_{i_j}\sigma_{i_j}) = x\sigma \). Hence \( x\sigma = x\sigma_{i_0} \), so that \( \sigma \in \bigcap (R(S_{i_j}) + \sigma_{i_j}) \), completing the proof.

If a ring \( E(F, A) \) is \( \nu \)-transitive for every ordinal \( \nu \) it must be the ring \( T(F, A) \) of all linear transformations of \((F, A)\). Hence the latter ring may be characterized as a ring satisfying (1) and (2) Any set of \( W \)-cosets with the finite intersection property has a nonvacuous intersection.

The condition (2) is essentially the same as (VII) used by Baer in [1].

If \( K \) is a topological ring it shall be called \emph{linearly \( \nu \)-compact} if (2) holds where \( W \)-cosets are replaced by cosets of closed right ideals. It is \emph{linearly compact} if linearly \( \nu \)-compact for all \( \nu \). (Cf. [5].)

The proof of the theorem implies that a primitive ring which is linearly compact in any topology which makes \( R(S) \) (for \( S \) a one-

\(^1\) This portion of the proof is due to the referee who kindly pointed out a slight gap in our original version.
dimensional subspace) closed must be the ring of all linear transformations of a vector space. This would be true in particular if maximal ideals were closed (\(R(S)\) is always maximal), or if the ring contained minimal ideals. For in the latter case \(R(S)\) is an annulet \[2, p. 19\] and thus closed in any topology making the ring a topological ring. Suppose \(K\) is a primitive and linearly compact Banach algebra. Then \(K\) is continuously isomorphic to a dense and linearly compact algebra of bounded operators in a Banach space. Since \(R(S)\) is closed, the algebra of operators contains all linear transformations of the space. But unless the space is finite-dimensional this would include transformations which are not continuous, an obvious contradiction.

The rings \(T(F, A)\) which occur as linearly compact primitive rings need not satisfy the minimum condition. For if \((F, A)\) is infinite-dimensional, the ring \(T_s(F, A)\) (the ring of all linear transformations of \((F, A)\) of rank less than \(\aleph_\alpha\)) is linearly \(\nu\)-compact in the finite topology. This can be seen as follows: In this topology all closed right ideals have the form \(R(T)\), for \(T\) a subspace of \((F, A)\) \[2, p. 20\]. Now assume \(\{R(S_i) + \sigma_i\}\) has rank less than \(\aleph_\alpha\) and possesses the finite intersection property. As in proof of the theorem, there exists \(\sigma \in T(F, A)\) such that \(\sigma = \sigma_i\) on each \(S_i\), so that \(\sigma \in \cap (R(S_i) + \sigma_i)\). Although \(S = \sum S_i\) is of arbitrary rank \(r\), we have

\[
r(S\sigma) = r(\sum S_i\sigma) \leq \sum r(S_i\sigma_i) < \aleph_r,
\]

since \(\sigma_i \in T_s(F, A)\), and therefore \(S\sigma_i\) has rank less than \(\aleph_r\). Put \(A = S \oplus Q\) and let \(\sigma'\) be that transformation in \(T(F, A)\) which agrees with \(\sigma\) on \(S\) and satisfies \(Q\sigma' = 0\). Then \(r(A\sigma') = r(S\sigma)\) so that \(\sigma' \in T_s(F, A)\), and since it agrees with \(\sigma\) on \(S\) we have \(\sigma' \in \cap (R(S_i) + \sigma_i)\), which completes the proof.

As a special case, we have that \(T(F, A)\) is linearly compact in the finite topology. It is of course an easy matter to construct examples of \(\nu\)-transitive rings which are not linearly \(\nu\)-compact in the finite topology.

**References**