FIXED POINTS OF PRODUCTS AND ORDERED SUMS OF SIMPLY ORDERED SETS

SEYMOUR GINSBURG

Let $A$ and $B$ be simply ordered sets and $A \times B$ the cartesian product ordered by first differences. Sufficiency conditions are given on the sets $A$ and $B$ for the existence of fixed points in the set $A \times B$ (Theorem 2). Sufficiency conditions are given on the sets $A$, $B$, and $C$ for the incomparability of the order types $|A \times B|$ and $|A \times C|$, ($|C \times A|$ and $|C \times B|$) (Corollary 2 of Theorem 7). The content of Theorem 4 is that if $\{A_\xi\}$ is a family of $2^\aleph_0$ linear sets, each containing a fixed point, then there exists a subset $G$ of $R \times R$, where $R$ denotes the real numbers, such that for each element $x$ in $R$, if $B_x = \{y \mid (x, y) \in G\}$, either $B_x$ is empty or is one of the sets $A_\xi$. Furthermore, if $p$ is a fixed point in $B_x$, then $(x, p)$ is a fixed point in $G$.

1. Fixed points of ordered sums and products. The element $p$ of the simply ordered set $A$ is said to be a fixed point of $A$ if $f(p) = p$ for every similarity transformation $f$ of $A$ into $A$.

**Lemma 1.** Let $p$ and $q$ be fixed points of the disjoint simply ordered sets $A$ and $B$ respectively. Then either $p$ or $q$ is a fixed point of the ordered sum, $C = A + B$.

**Proof.** Assume the contrary, i.e., assume that there exist similarity transformations $f$ and $g$ of $C$ into $C$ for which $f(p) \neq p$ and $g(q) \neq q$. Suppose that $f(p) < p$. Then the function $h$, which is defined by $h(x) = x$ for $x > p$, $x$ in $A$, and $h(x) = f(x)$ for $x \leq p$, is a similarity transformation of $A$ into $A$ for which $h(p) \neq p$. From this contradiction of $p$ being a fixed point in $A$, we see that $f(p) > p$. Analogous arguments show that $g(q) < q$, $f(q) \leq q$, and $g(p) \geq p$. Let $k$ be the similarity transformation of $C$ into $C$ which is defined by $k(x) = f(x)$ for $x < q$ and $k(x) = x$ for $x \geq q$. This shows that without loss of generality we

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Presented to the Society, September 5, 1952 under the title, A theorem on similarity transformations, and November 29, 1952 under the title Some results on fixed points of simply ordered sets; received by the editors December 18, 1952 and, in revised form, October 23, 1953.

1 If $A$ is a simply ordered set, $|A|$ denotes its order type.

2 Let $B$ be a simply ordered set. If $\{A_b \mid b \in B\}$ is a family of pairwise disjoint simply ordered sets, then the ordered sum, $\sum_{b \in B} A_b$, is the set union of the $A_b$, the elements being ordered as follows. If $c$ and $d$ are two distinct elements in $A_b$, then $c$ and $d$, considered in the ordered sum, have the same order as in the set $A_b$. If $c$ is in $A_a$ and $d$ is in $A_b$, where $a < b$, then $c < d$. 

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may suppose that \( f(x) = x \) for \( x \geq q \), and likewise, that \( g(x) = x \) for \( x \leq p \).

Suppose that \( fg(B) \) is a subset of \( B \). As \( q \) is a fixed point in \( B \), 
\( fg(q) = q \). On the other hand, since \( g(q) < q \), it follows that \( fg(q) < f(q) = q \). From this contradiction we conclude that \( fg(B) \cap A \) is nonempty, and thus that \( fg(A) \) is a proper subset of \( A \).

Since \( fg \), acting on \( A \), is a similarity transformation of \( A \) into \( A \), 
\( fg(p) = p \). But \( f[g(p)] = f(p) > p \). This is a contradiction. Thus either \( p \) or \( q \) is a fixed point in \( C \). Q.E.D.

**Remark.** It is not true, in general, that both \( p \) and \( q \) are fixed points in \( C \). One need only consider the case where \( A = \{ \xi \mid \xi \leq \omega \} \), \( B = \{ \xi \mid \omega < \xi \leq \omega + \omega \} \), \( p = \omega \), and \( q = \omega + \omega \).

**Corollary.** If \( \{ A_1, A_2, \ldots, A_n \} \) are a family of pairwise disjoint sets, and \( p_j \) is a fixed point in \( A_j \), then at least one of the points \( p_j \) is a fixed point in the ordered sum, \( C = A_1 + \cdots + A_n \).

**Theorem 1.** Let \( \{ A_1, A_2, \ldots, A_n \} \) be a family of pairwise disjoint sets such that \(^3 |A_i| = |A_j| \) for \( i, j \leq n \). For \( i = 2, 3, \ldots, n \), let \( f_i \) be a similarity transformation of \( A_1 \) into \( A_i \). Let \( p \) be a fixed point of \( A_1 \), and \( B \) the ordered sum, \( A_1 + A_2 + \cdots + A_n \). (a) If \( p \) is a fixed point in \( A_1 \cup A_2 \), then \( p \) is a fixed point in \( B \), and thus a fixed point in \( \bigcup_{i=1}^{n} A_i \). If \( f_2(p) \) is a fixed point in \( A_1 \cup A_2 \), then \( f_n(p) \) is a fixed point in \( B \). (b) If both \( p \) and \( f_2(p) \) are fixed points in \( A_1 \cup A_2 \), then each point \( f_i(p) = p_i \) is a fixed point in \( B \).

**Proof.** Using induction suppose that the theorem has been demonstrated for \( n = 2, 3, \ldots, k \).

To prove the first sentence in (a) suppose that \( p \) is not a fixed point in the set \( B = \bigcup_{i \leq k+1} A_i \), i.e., there is a similarity transformation \( f \) of \( B \) into \( B \) for which \( f(p) \neq p \). Let \( m \) be the largest integer \( \leq k \) for which \( f(p_j) \neq p_j \) for \( j \leq m \). If \( m < k \), then \( f(p_{m+1}) = p_{m+1} \). Thus the function \( g \) on \( C = \bigcup_{i \leq k} A_i \), which is defined by \( g(x) = f(x) \) for \( x < p_{m+1} \) and \( g(x) = x \) for \( x \geq p_{m+1} \), is a similarity transformation. Since \( g(p) \neq p \), our induction hypothesis that \( p \) is a fixed point in \( C \) is contradicted. Now suppose that \( m = k \). As is easily seen from our induction hypothesis, \( f(p_m) > p_m \). Let \( D = A_k \cup A_{k+1} \). Let \( h \) be the similarity transformation of \( D \) into \( D \) which is defined by \( g(x) = x \) for \( x < p_m \), and \( g(x) = f(x) \) for \( x \geq p_m \). Since \( |A_i| = |A_j| \), from I of §2 of [4], \( p_m \) is a fixed point in \( D \). But \( g(p_m) \neq p_m \). From this contradiction we see that \( p \) must be a fixed point in \( B \).

\(^3 |A_i| = |A_j| \) if there exists a similarity transformation of \( A_i \) into \( A_j \), and a similarity transformation of \( A_j \) into \( A_i \).
To see the second sentence in (a) let \(g_2\) be a similarity transformation of \(A_2\) into \(A_1\). Since \(g_2f_2\) is a similarity transformation of \(A_1\) into \(A_1\) and \(p\) is a fixed point in \(A_1\), \(g_2[f_2(p)] = p\). Let \(\sigma\) be the similarity transformation of \(A_1 \cup A_2\) into \(A_k \cup A_{k+1}\) which is defined by \(\sigma(x) = f_k(x)\) for \(x\) in \(A_1\) and \(\sigma(x) = f_{k+1}g_2(x)\) for \(x\) in \(A_2\). Since \(|A_1 \cup A_2| = |A_k \cup A_{k+1}|\), by I of §2 of [4], \(\sigma[f_2(p)] = f_{k+1}(p)\) is a fixed point in \(A_k \cup A_{k+1}\). By a method analogous to that in the preceding paragraph one shows that \(p_{k+1}\) is fixed in \(A_{k-1} \cup A_k \cup A_{k+1}\), then fixed in \(A_{k-2} \cup A_{k-1} \cup A_k \cup A_{k+1}\), etc, and finally, fixed in \(U_{j\leq k+1}A_j\).

To see part (b) suppose that both \(p\) and \(p_2\) are fixed points in the set \(A_1 \cup A_2\). Furthermore, suppose that for some \(j\), there exists a similarity transformation \(f\) of \(B\) into \(B\) for which \(f(p_j) \neq p_j\). Clearly \(j < n\). If \(f(p_j) < p_j\), then it is easy to construct a similarity transformation \(g\) of \(U_{i\leq j}A_i\) into \(U_{i\leq j}A_i\) for which \(g(p_j) \neq p_j\). This contradicts part (a) for \(n = j\). If \(f(p_j) > p_j\), then there exists a similarity transformation \(h\) of \(U_{j\leq i\leq n}A_i\) into \(U_{j\leq i\leq n}A_i\) for which \(h(p_j) \neq p_j\). By proceeding as in the previous paragraph we see that \(p_j\) is fixed in \(A_j \cup A_{j+1}\). On applying (a) we see that \(p_j\) is fixed in \(U_{j\leq i\leq n}A_i\). Since \(h(p_j) \neq p_j\) we have a contradiction. Therefore \(f(p_j) = p_j\). Q.E.D.

We now consider the existence of fixed points in the set \(A \times B\), where \(A \times B\) is ordered by first differences, i.e., \((a, b) < (c, d)\) if \(a < c\), or if \(a = c\) and \(b < d\). It is obvious that if either \(A\) or \(B\) contains no fixed points, then \(A \times B\) contains none. The question arises as to what occurs in the case where \(A\) and \(B\) both contain fixed points.

**Theorem 2.** Let \(p\) and \(q\) be fixed points respectively of the simply ordered sets \(A\) and \(B\). Then there exists a fixed point \((r, q)\) of \(A \times B\), where \(r\) is some fixed point of \(A\) such that both of the sets

\[
\{x \mid r \leq x \leq p, \ x \in A\} \quad \text{and} \quad \{x \mid p \leq x \leq r, \ x \in A\}
\]

are finite.

**Proof.** For two points \(a\) and \(b\) in \(A\) let \(aRb\) if both of the sets

\[
\{x \mid a \leq x \leq b, \ x \in A\} \quad \text{and} \quad \{x \mid b \leq x \leq a, \ x \in A\}
\]

are finite. Let \(N_a\) be the equivalence class, determined by the equivalence relation \(R\), which contains the element \(a\). The order type of the set \(N_a\) is either \(\omega\), \(\omega^*\), \(\omega^*+\omega\), or \(n\), where \(n\) is finite.

Now let \(f\) be a similarity transformation of \(A \times B\) into \(A \times B\). Suppose that the order type of the set \(N_a\) is \(\omega\), i.e., the elements of \(N_a\) are \(y_1 < y_2 < \ldots < y_n < \ldots\). Let \(E\) be the set of \(A\) coordinates of the elements of the set \(f(N_a \times B)\). Since \(q\) is a fixed point of \(B\), by Lemma
1.3 of [4], $|m \times B| < |(m+1) \times B|$ for each positive integer $m$. Thus the set $E$ is infinite. Let $z_1 < z_2 < \cdots < z_n < \cdots$ be a sequence of points of $E$. Let $g(y_n) = z_{4n+2}$ if $y_n \neq z_{4n+3}$, and $g(y_n) = z_{4n+3}$ if $y_n = z_{4n+2}$. If the order type of the set $N_a$ is $\omega^*$, or $\omega^* + \omega$, then $g$ can be defined analogously to the above, so that $g(x) \neq x$ for each $x$ in $N_a$. Finally, suppose that the order type of the set $N_a$ is $n$, where $n$ is finite. Again let $E$ be the set of $A$ coordinates of elements in the set $f(N_a \times B)$. Let $F = \{ x \mid c < x < d, c \in E, d \in E, x \in A \}$ and $G = E \cup F$. Since $|B| |j| < |B| |(j+1)|$ the power of the set $G$ is at least $n$. If the power of the set $G$ is greater than $n$, then a similarity transformation $g$, of $N_a$ into $G$, can be defined in such a manner that $g(x) \neq x$. If the power of $G$ is exactly $n$, then a similarity transformation $g$, from $N_a$ to $G$, can be defined in precisely one way.

The function $g$ is thus defined to be one-to-one on each set $N_a$, and as is easily seen, is also one-to-one on all of $A$. Clearly $g$ preserves the order of the elements. Thus $g$ is a similarity transformation of $A$ into $A$. As $p$ is a fixed point of $A$, $g(p) = p$. This implies that $N_p$ is finite and the set $G$ that is associated with $N_p$ contains exactly as many elements as $N_p$. Since each element $y$ in $N_p$ is also a fixed point in $A$ (IV of §2 of [4]), $g(y) = y$. This shows that $f(N_p \times B)$ is a subset of $N_p \times B$. This is also true for any other similarity transformation $h$ of $A \times B$ into $A \times B$. Since $N_p \times B$ is the ordered sum of $n$ disjoint sets which are similar to $B$, $n$ being the number of elements in $N_p$, by the Corollary to Lemma 1, one of the points $(r, q)$, where $r$ is some element in $N_p$, is a fixed point. Q.E.D.

If each point of $B$ is a fixed point of $B$, then the conclusion of Theorem 2 can be strengthened. First, however, we prove a general result on similarity transformations.

**Lemma 2.** Let $g$ be a similarity transformation of $C \times D$ into $E \times F$ which has the property that for each element $a$ in $C$, there are at least two elements $u(a)$ and $v(a)$, $u(a) < v(a)$, in $E$ such that $g(a \times D) \cap (u(a) \times F)$ $\neq \emptyset$ and $g(a \times D) \cap (v(a) \times F) \neq \emptyset$.

Then (a) for a given element $p$ in $C$ and a given element $q$ in $E$, where $u(p) \leq q \leq v(p)$, a similarity transformation $h$ of $C$ into $E$ can be found so that $h(p) \neq q$, and for each element $a$ in $C$, either $h(a) = u(a)$ or $h(a) = v(a)$; and (b) if $C = E$, then $h$ can be chosen so that $h(a) \neq a$ and $u(a) \leq h(a) \leq v(a)$.

**Proof.** Let $[u(a), v(a)] = \{ x \mid u(a) \leq x \leq v(a), x \in E \}$, and $W$ if there exists a similarity transformation of $A$ into $B$, but no similarity transformation of $B$ into $A$. See [7, p. 253].
\[ \{ [u(a), v(a)] | a \in C \}. \]

Since \( g \) is a similarity transformation, it follows that no element of \( E \) can be in more than two elements of \( W \). Furthermore, if the element \( x \) is in two elements of \( W \), say \( w_1 \) and \( w_2 \), then \( w_1 = [a, v] \) and \( w_2 = [z, a] \), or \( w_1 = [z, a] \) and \( w_2 = [a, v] \). Clearly if \( a < b \), then \( u(a) < v(a) \leq u(b) < v(b) \).

Suppose that the two elements, \( p \) in \( C \) and \( q \) in \( E \), \( u(p) \leq q \leq v(p) \), are given. If \( u(p) \neq q \), let \( h(x) = u(x) \) for \( x \) in \( C \). If \( u(p) = q \), let \( h(x) = v(x) \) for \( x \) in \( C \). The function \( h \) satisfies the conclusions in (a).

Now suppose that \( C = E \). For any two elements \( a \) and \( b \) in \( E \), let \( aRb \) and \( N_a \) have the same meaning as in Theorem 2. Suppose that the order type of \( N_a \) is \( \omega \), the elements of \( N_a \) being \( y_0 < y_1 < \cdots \). It follows from the definition of the set \( N_a \) and the fact that the \( y_n \) are a sequence of consecutive elements in \( E \) that \( [u(y), v(y)] \cap [u(y_n), v(y_n)] = \emptyset \) for any element \( y \) in \( E - N_a \). Let \( h(y_0) = u(y_0) \) if \( u(y_0) \neq y_0 \), and \( h(y_0) = v(y_0) \) if \( u(y_0) = y_0 \). Suppose that \( h(y_i) \) has been defined for \( i \leq j \). Two possibilities occur. Either \( v(y_j) < u(y_{j+1}) \) or \( v(y_j) = u(y_{j+1}) \). Consider the first alternative. Let \( h(y_{j+1}) = u(y_{j+1}) \) if \( u(y_{j+1}) \neq y_{j+1} \), and \( h(y_{j+1}) = v(y_{j+1}) \) if \( u(y_{j+1}) = y_{j+1} \). Consider the second alternative. If \( v(y_{j+1}) \neq y_{j+1} \), let \( h(y_{j+1}) = v(y_{j+1}) \). Suppose that \( v(y_{j+1}) = y_{j+1} \). As \( y_j \) and \( y_{j+1} \) are consecutive elements in \( E \), \( v(y_j) \leq y_j \). If \( v(y_j) = y_j \), then from our induction hypothesis that \( h(y_j) \neq y_j \), and \( h(y_j) \leq v(y_j) \), we see that \( h(y_j) < y_j \). In this case, let \( h(y_{j+1}) = u(y_{j+1}) = y_j \). If \( v(y_j) < y_j \), let \( h(y_{j+1}) = y_j \). Note that \( u(y_{j+1}) \leq h(y_{j+1}) \leq v(y_{j+1}) \). In this way the function \( h \) is defined on \( N_a \). If the order type of \( N_a \) is \( \omega * \), \( \omega * + \omega \), or \( n \), then an analogous argument is applicable. The function \( h \) is thus defined on each set \( N_a \), and hence on all of \( E \). Clearly it is a similarity transformation and satisfies the conclusions in (b). Q.E.D.

A simply ordered set \( E \) is called exact if \( E \) is nonempty and if each point of \( E \) is a fixed point.

**Theorem 3.** Let \( p \) be a fixed point of \( E \) and let \( F \) be an exact set. Then for each element \( y \) in \( F \), \( (p, y) \) is a fixed point of \( E \times F \).

**Proof.** Let \( g \) be any similarity transformation of \( E \times F \) into itself. Let \( a \) be any element of \( E \) for which there exists an element \( b \) such that \( g(a \times F) \subseteq (b \times F) \). For each element \( y \) in \( F \), let \( m(y) = z \), where \( g([(a, y)]) = (b, z) \). The function \( m(y) \) is a similarity transformation of \( F \) into \( F \). As \( F \) is exact, \( m(y) = y \) for each element \( y \) in \( F \), i.e., \( g([(a, y)]) = (b, y) \). Therefore \( g(a \times F) = (b \times F) \).

Suppose that \( g(p \times F) \) is not the set \( (p \times F) \). Let

\[ B = \{ a | a \in E, g(a \times F) = (b_a \times F) \} \]

and

\[ G = \{ b_a | a \in B \}. \]
Let $H = E - B$ and $M = E - G$. Since $g(a \times F) = (b_a \times F)$ for each element $a$ in $B$, it follows that $g(H \times F) \subseteq (M \times F)$. Furthermore, for each element $y$ in $H$, there are two elements, $u(y)$ and $v(y)$, $u(y) < v(y)$, in $M$ such that

$$g(y \times F) \cap (u(y) \times F) \neq \emptyset \quad \text{and} \quad g(y \times F) \cap (v(y) \times F) \neq \emptyset.$$ 

Two possibilities arise. Either $p$ is in $B$ or $p$ is in $H$. Suppose that the first alternative occurs. Then $g(p \times F) = (b_p \times F)$, where $p \neq b_p$. An application of Lemma 2 yields a similarity transformation $h$ of $H$ into $M$ such that $u(x) \leq h(x) \leq v(x)$, and $h(x) \neq x$. Let $f$ be the function which is defined by $f(x) = h(x)$ for each element $x$ in $H$, and $f(a) = b_a$ for each element $a$ in $B$. The function $f$ is a similarity transformation of $E$ into $E$ for which $f(p) \neq p$. This is a contradiction of the fact that $p$ is a fixed point of $E$. Suppose that the second possibility occurs, i.e., $p$ is in $H$. Applying Lemma 2 we obtain a function $h$ of $H$ into $M$ for which $h(p) \neq p$, and $u(x) \leq h(x) \leq v(x)$ for each element $x$ in $H$. Let $f(x) = h(x)$ for $x$ in $H$ and $f(a) = b_a$ for $a$ in $B$. Again $f(p) \neq p$. Thus both alternatives lead to a contradiction. Consequently we must have $g(p \times F) = (p \times F)$, i.e., $g([p, y]) = (p, y)$ for each element $y$ in $F$. Therefore $(p, y)$ is a fixed point of $E \times F$.

**Corollary.** If $E$ and $F$ are both exact, then $E \times F$ is exact.

2. **Imbedding sets.** The set of real numbers, ordered in the natural manner, is denoted by $R$. A subset of $R$ is called a linear set. By $\theta$ is meant the smallest ordinal number whose power is $2^{\aleph_0}$.

Let $A_0$ be a given linear exact set of power $2^{\aleph_0}$. For each ordinal number $\xi$ where $1 \leq \xi < \theta$ let $A_\xi$ be a linear set which is similar to $A_0$. Well order the elements of $A_0$ into the sequence $\{x_\xi\}_{\xi < \theta}$ and let $G = \bigcup_{\xi < \theta} (x_\xi \times A_\xi)$. Consider the set $G$ so obtained. $G$ is a subset of $R \times R$. The order type of $G$ is $A_0 \times A_0$, and so is exact by the corollary to Theorem 3. Furthermore, for each element $x$ in $R$, if $B_x = \{y | (x, y) \in G\}$, then either $B_x$ is empty or else is one of the sets $A_\xi$. This suggests the question of whether or not there exists an exact subset $G$ of $R \times R$, which has the property that the family of sets $\{B_x | x \in R\}$ contains all linear exact sets. No such set $G$ can exist, however, since there are $2^{2^{\aleph_0}}$ exact linear sets and only $2^{\aleph_0}$ sets $B_x$. Now if we only demand that the family of sets $\{B_x\}$ contain a pre-assigned family of $2^{\aleph_0}$ exact linear sets, then the set $G$ does exist. More precisely we have

**Theorem 4.** Let $\{A_\xi | \xi < \theta\}$ be a family of nonempty linear sets $A_\xi$ such that for each $\xi$, $|A_\xi| < |R|$. Then there exists a subset $G$ of $R \times R$ with the following properties:
(a) For each element $x$ in $R$, the set $B_x = \{ y \mid (x, y) \in G \}$ is either empty or else is one of the sets $A_x$.

(b) For each set $A_x$ there is only one element $x$ in $R$ such that $B_x = A_x$.

(c) The set $T = \{ x \mid B_x \neq \emptyset \}$ is exact.

(d) If $p$ is a fixed point in $B_x$, then $(x, p)$ is a fixed point in $G$.

**Proof.** The demonstration is divided into two cases.

(a) Suppose that only a denumerable number of the sets, say $\{ A_n \mid n < \gamma \leq \omega \}$, are of power $\geq 2$ each. By (a, b) is meant the open interval $\{ x \mid a < x < b \}$ of $R$. Let $S$ denote the set of positive real numbers. For each $n < \gamma$ let $C_n$ be a subset of $(2n, 2n+1)$ which is similar to $A_n$. Let $f$ be a similarity transformation of $S$ into $S$ which is not the identity function on $(2n+1, 2n+2)$ for some $n < \gamma$. Let $F$ be the family of all such functions $f$. The power of $F$ is $2^{\aleph_0}$. Denote by $f^*$ the inverse function of $f$. Well order the elements of $S$ and $F$ into the two sequences, $\{ y_\xi \}_{\xi < \theta}$ and $\{ f_\xi \}_{\xi < \theta}$, respectively. For each $\xi$ let

$$D_\xi = \{ x \mid f_\xi(x) \neq x, x \in (2n+1, 2n+2) \}$$

and $D_\xi = \bigcup_{n < \gamma} D_\xi^n$. Clearly each set $D_\xi$ contains some open interval of $S$, call it $J_\xi$. Since $|A_n| = |C_n| < |R|$, by Theorem 8 of [7], $\sum_{n < \gamma} |C_n| < |R|$. Let $Z = \bigcup_{n < \gamma} C_n$, $p_0$ be the first element in the set $J_0 = f_0^*(Z)$, and $q_0 = f_0(p_0)$. The element $p_0$ certainly exists since, by Lemmas 3 and 4 of [7], if $|A| < |R|$ and $|B| = |R|$, then the power of the set $B = A$ is $2^{\aleph_0}$. Now suppose that $\{ p_\xi \}_{\xi < \mu}$ and $\{ q_\xi \}_{\xi < \mu}$ where $\mu < \theta$ have been defined. If $G_\mu = Z \cup E_\mu$, where

$$E_\mu = \{ p_\xi \mid \xi < \mu \} \cup \{ q_\xi \mid \xi < \mu \},$$

then by Lemma 3 of [7], $|f_\mu^*(G_\mu) \cup E_\mu| < |R|$. Let $p_\mu$ be the first element in the set $J_\mu = f_\mu^*(G_\mu) \cup E_\mu$ and $q_\mu = f_\mu(p_\mu)$. Let $Y = Z \cup \{ p_\xi \mid \xi < \theta \}$.

Suppose that $p$ is either a fixed point in $C_i$ or $p$ is a point in $Y - Z$, which is not a fixed point in $Y$. Then there exists a similarity transformation $g$ of $Y$ into $Y$ such that $g(p) \neq p$. Either $p$ is in some set $C_i$, where $i < \gamma$, or else $p$ is in some set $(2n+1, 2n+2)$, where $n < \gamma$. Assume that the former is the situation. Since $p$ is a fixed point of $C_i$, $g(C_i)$ is not a subset of $C_i$. This implies that there exists an element $q$ in $Y$ such that $g(q) \neq q$ and $q$ is in either $(2i-1, 2i)$ or $(2i+1, 2i+2)$. From this we see that $g$ is not the identity function on $Y \cap (\bigcup_{n < \gamma} (2n+1, 2n+2))$. By a familiar argument, such as given in Theorem 3 of [2] and Theorem 2.4 in [4], it can be shown that each set $Y \cap (2n+1, 2n+2)$, for $n < \gamma$, is a dense exact subset of $(2n+1, 2n+2)$. From this it follows that the function $g$ can be extended to become an element $f_\gamma$ of $F$. Therefore the element $g(p_\mu) = q_\mu$ is not in
Y, so that \( g \) does not map \( Y \) into \( Y \). From this contradiction we conclude that the point \( p \) is fixed in \( Y \).

A similar argument shows that if \( W = \{ p_{\xi} | \xi < \theta \} \cup \{ w_n | n < \gamma \} \), where \( w_n \) is some definite point in \( C_n \), then each point of \( W \) is fixed, i.e., \( W \) is exact.

Now the power of the set \( Y - Z \) is \( 2^{\aleph_0} \). Well order the elements of the set \( Y - Z \) into the sequence \( \{ x_{\xi} \}_{\gamma \leq \xi < \theta} \). Denote by \( G \) the set

\[
G = \bigcup_{n<\gamma} \left[ (2n + \frac{1}{2}) \times A_n \right] \cap \{ (x_{\xi} \times A_\xi) | \gamma \leq \xi < \theta \}.
\]

Evidently \( |G| = |Y| \) since each set \( A_\xi, \xi \geq \gamma \), consists of one point. It is easy to verify that \( G \) satisfies the conclusions of the theorem.

(b) Suppose that a nondenumerable number of the sets \( A_\xi \) are of power \( \geq 2 \) each. Let \( C \) be a dense exact subset of \( R \), of power \( 2^{\aleph_0} \), which has the property that no two disjoint subsets of \( C \), of power \( 2^{\aleph_0} \) each, are similar. Furthermore, let each point of \( C \) be a \( c \)-condensation point of \( C \), where by \( x \) being a \( c \)-condensation point of \( C \) is meant that each open interval of \( R \), \((a, x)\) and \((x, b)\), meets the set \( C \) in \( 2^{\aleph_0} \) points. The existence of such a set \( C \) is guaranteed by Lemma 1 and Theorems 4 and 5 of [3]. Divide the family of sets \( \{ A_\xi | \xi < \theta \} \) into two disjoint families, \( \{ D_\xi | \xi < \mu \leq \theta \} \) and \( \{ C_\xi | \xi < \theta \} \), where the first family is a nondenumerable number of sets, of power \( \geq 2 \) each. As is well known the elements of \( C \) can be divided into two disjoint sets, \( M = \{ u_\xi | \xi \leq \mu \} \) and \( N = \{ v_\nu | \nu < \theta \} \), such that \( M \) is dense in \( C \), and each element \( u_\xi \) is a condensation point of \( C \) (in the usual sense).

Now denote by \( G \) the set

\[
G = \bigcup_{\xi \leq \mu} (u_\xi \times D_\xi) \cup \bigcup_{\nu < \theta} (v_\nu \times C_\nu).
\]

The only property of \( G \) that is not self-evident is (d). To see that (d) is satisfied suppose the contrary, i.e., suppose that there exists a similarity transformation of \( G \) into \( G \) such that \( f(p) \neq p \) for some element \( p \) in \( G \), where \( p = (d, t) \) and \( t \) is a fixed point in \( B_d \). Since \( t \) is a fixed point in \( B_d \), it follows that for some element \( (d, q) \) in the set \((d, B_d)\), the element of \( f[(d, q)] \) is not in \((d \times B_d)\), i.e., \( f[(d, q)] = (z, r) \) where \( d \neq z \). Suppose that \( z > d \). Denote by \( P \) the set \( P = \{ x | d < x < z, x \in C \} \). Note that \( f[(x, y)] > (x, y) \) for \((x, y)\) in \( G \), where \( x \) is in \( P \). Since each element of \( C \) is a \( c \)-condensation point of \( C \), the power of \( P \) is \( 2^{\aleph_0} \). Suppose that for two elements \( x_1 \) and \( x_2 \), \( x_1 < x_2 \), in \( P \), there exists an element \( x_3 \) in \( C \) such that

\[
f(x_1 \times B_{x_1}) \cap (x_3 \times B_{x_2}) \neq \emptyset \quad \text{and} \quad f(x_2 \times B_{x_2}) \cap (x_3 \times B_{x_2}) \neq \emptyset.
\]

This implies that for each element \( u_\xi \) in \( M \cap P \), \( f(u_\xi \times D_\xi) \subseteq (x_3 \times B_{x_2}) \).
Now there are a nondenumerable number of such elements \( u_\xi \), from the selection of the \( u_\xi \) and \( v_\xi \). Since each set \( D_\xi \) contains at least two elements, it follows that the set \( (x_3 \times B_{x_3}) \) contains a nondenumerable number of disjoint intervals. But this is impossible since \( (x_3 \times B_{x_3}) \) is similar to a subset of \( R \). Consequently, for any two elements \( x_1 \) and \( x_2 \) in \( P, f(x_1 \times B_{x_1}) \) and \( f(x_2 \times B_{x_2}) \) cannot meet the same set \( (x_3 \times B_{x_3}) \). This fact implies the following:

For each subset \( (x \times B_x) \) of \( G \), where \( x \) is in \( P \), select a definite element \( w_x \) in \( B_x \). Let \( g \) be the function which is defined by \( g(x) = y \), where \( f([(x, w_x)]) = (y, t) \). Then \( g \) is a similarity transformation of \( P \) into \( C-P \).

From this we obtain two similar, disjoint subsets of \( C \), of power \( 2^{N_0} \) each, \( P \) and \( g(P) \). This, in turn, contradicts the fact that \( C \) does not contain two such sets. Therefore \( f([(d, q)]) > (d, q) \) is false. An analogous argument shows that \( f([(d, q)]) < (d, q) \) is also false. Hence we are forced to conclude that no such function \( f \) can exist, i.e., the original point \( p \) is fixed. Q.E.D.

**Corollary 1.** Let \( \{A_\xi\}_{\xi < \theta} \) be a sequence of nonempty linear sets such that \( |A_\xi| < |R| \), for each \( \xi \). Then there exists a subset \( G \) of \( R \times R \) which satisfies (a), (c), and (d) of Theorem 4, and for which there exists a one-to-one correspondence \( h \) between the sets \( \{\xi | \xi < \theta\} \) and the set \( \{x | B_x \neq \emptyset\} \) such that \( A_\xi = B_{h(\xi)} \).

**Corollary 2.** Let \( \{A_\xi | \xi < \mu \leq \theta\} \) be a family of nonempty linear sets such that \( |A_\xi| < |R| \) for each \( \xi \). Then there exists a subset \( G \) of \( R \times R \) which satisfies (a), (c), and (d) of Theorem 4.

If each set \( A_\xi \) contains at least one fixed point, then \( |A_\xi| < |R| \). If each set \( A_\xi \) is exact we obtain

**Corollary 3.** Let \( \{A_\xi | \xi < \theta\} \) be a family of linear exact sets. Then there exists an exact subset \( G \) of \( R \times R \) satisfying (a), (b), and (c) of Theorem 4.

The assumption in Theorem 4 that \( |A_\xi| < |R| \) for each \( \xi \) cannot be removed. For example, let \( A_0 = R \) and \( A_\xi = \{p_\xi\}, \xi \geq 1, \) where \( p_\xi \neq p_\rho \) for \( \xi \neq \rho \). Any subset \( G \) of \( R \times R \) satisfying (a) and (b) is of order type \( \equiv |R| \), thus contains no fixed point, and so violates (d).

The conclusion of Corollary 2 cannot be strengthened to include (b). This is seen by the following example. Let \( A_0 \) be a linear set containing a fixed point. For each positive integer \( i \) let \( A_i \) be a linear set, similar to \( A_0 \), such that \( A_m \neq A_n \) for \( m \neq n \). Now suppose that a subset \( G \) of \( R \times R \) exists which satisfies (a) and (b). Then the set \( T \) is denumerably infinite. But in Theorem 1 of [2] Dushnik and Miller have
demonstrated that there is no denumerably infinite exact set. Thus (c) and (d) are not satisfied.

3. Related results. In this section some applications of Lemma 2 are given.

A simply ordered set $E$ is said to have the fixed point property if, for each similarity transformation $f$ of $E$ into itself, there exists an element $p$ of $E$ such that $f(p) = p$.

The ordered couple of simply ordered sets $(E, F)$ is called an $A$-pair if, for each pair of simply ordered sets $G$ and $H$, and similarity transformation $f$ of $E \times G$ into $F \times H$, there exist two elements, $p$ in $E$ and $q$ in $F$, such that $f(p \times G)$ is a subset of $(q \times H)$ [1].

From Lemma 2 there immediately follows

**Theorem 5.** If $A$ and $B$ are any two sets for which there is no similarity transformation of $A$ into $B$, then $(A, B)$ is an $A$-pair.

Turning to sets $E$ for which $(E, E)$ is an $A$-pair we have

**Theorem 6.** If $E$ has the fixed point property, then $(E, E)$ is an $A$-pair.

**Proof.** Let $g$ be a similarity transformation of $E \times D$ into $E \times F$, and suppose that there are no two elements $a$ and $b$ in $E$ for which $g(a \times D)$ is a subset of $(b \times F)$. This means that for each element $a$ in $E$ there are two elements, $u(a)$ and $v(a)$, $u(a) < v(a)$, such that

$$g(a \times D) \cap (u(a) \times F) \neq \emptyset \quad \text{and} \quad g(a \times D) \cap (v(a) \times F) \neq \emptyset.$$  

Applying Lemma 2 we obtain a similarity transformation $h$ of $E$ into $E$ such that for each element $a$ in $E$, $h(a) \neq a$. This contradicts our assumption that $E$ has the fixed point property. It therefore follows that there must exist two elements $a$ and $b$ in $E$ such that $g(a \times D)$ is a subset of $(b \times F)$, i.e., $(E, E)$ is an $A$-pair.

**Corollary 1.** If $E$ has a fixed point, then $(E, E)$ is an $A$-pair.

Let $E$ be a well ordered set of order type $\alpha$. Arens has shown that if $\alpha = \omega$, then $(E, E)$ is not an $A$-pair [1]. We now extend that result. We may suppose that $E$ is the set of ordinal numbers which are smaller than $\alpha$. Suppose that $\alpha$ is a limit number. Consider the mapping $f$ of $E \times \{0, 1\}$ onto $E \times \{0\}$ which is defined as follows. For $\xi < \alpha$ and $\xi = \sigma_\xi + n_\xi$, where $\sigma_\xi = 0$ or is a limit number, and $n_\xi$ is a finite integer, let $f[(\xi, 0)] = (\sigma_\xi + 2n_\xi, 0)$ and $f[(\xi, 1)] = (\sigma_\xi + 2n_\xi + 1, 0)$. As $\alpha$ is a limit number $f$ is a similarity transformation of $E \times \{0, 1\}$ onto $E \times \{0\}$. As $f$ is one-to-one, there are no two elements $a$ and $b$ in $E$ for which $f(a \times \{0, 1\})$ is a subset of $(b, 0)$. Thus $(E, E)$ is not
an $A$-pair. Now suppose that $\alpha$ is a nonlimit number. Then the hypotheses of Corollary 1 of Theorem 6 are satisfied, so that $(E, E)$ is an $A$-pair. Summarizing the preceding discussion we get

**Corollary 2.** Let $E$ be a well ordered set of order type $\alpha$. Then $(E, E)$ is an $A$-pair if and only if $\alpha$ is not a limit number.

$|A|$ and $|B|$ are said to be incomparable if there is no similarity transformation of $A$ into $B$ and no similarity transformation of $B$ into $A$. It is easy to find examples in which $|A|$ and $|B|$ are incomparable, while for some nonempty set $C$, $|A \times C|$ and $|B \times C|$ (where $|C \times A|$ and $|C \times B|$) are not. This is so in the case where $|A| = \omega$, $|B| = \omega^*$, and $|C| = \eta$, $\eta$ being the order type of the rational numbers ordered in the natural manner. In fact, for each pair of nonempty sets $A$ and $B$, there always exists a set $C$ such that $|A \times C| = |B \times C|$ (where $|C \times A|$ and $|C \times B|$). For let $|D| = |A| + |B|$ and $|C| = |D|^*$ [5]. Then $|C| = |C|$ and $|C| = |A \times C| = |B \times C|$ (where $|C| = |C \times A|$ and $|C \times B|$).

**Theorem 7.** Let $A$ and $B$ be any two sets for which there is no similarity transformation of $A$ into $B$. Then

(a) for any set $C$ which has the fixed point property, there is no similarity transformation of $C \times A$ into $C \times B$;

(b) for any set $C$ such that $|C| < |C|^2$, there is no similarity transformation of $A \times C$ into $B \times C$.

**Proof.** (a) Suppose that $f$ is a similarity transformation of $C \times A$ into $C \times B$. For each element $c$ in $C$, $f(c \times A)$ is not a subset of $(d \times B)$, for some element $d$ in $B$. This is so since there is no similarity transformation of $A$ into $B$. Thus there are at least two elements in $B, u(c)$ and $v(c)$, with $u(c) < v(c)$, such that

$$f(c \times A) \cap (u(c) \times B) \neq \emptyset \quad \text{and} \quad f(c \times A) \cap (u(c) \times B) \neq \emptyset.$$ 

By Lemma 2, a similarity transformation $h$ of $C$ into $C$ can be found so that $h(y) \neq y$ for each element $y$ in $C$. This contradicts the fact that the set $C$ has the fixed point property. Therefore no such similarity transformation $f$ can exist.

(b) Since $|C| < |C|^2$, it follows from Lemma 1.3 of [4] that $|C| m < |C| (m + 1)$ for each positive integer $m$. Using this fact, it follows from an argument analogous to that given in Theorem 2 that if a similarity transformation $f$ of $A \times C$ into $B \times C$ existed, then a similarity transformation $h$ of $A$ into $B$ could be constructed. By hypothesis though, no such function $h$ exists. Hence our result. Q.E.D.

**Corollary 1.** If $|A| < |B|$ and $C$ has the fixed point property, then
\[ |C \times A| < |C \times B|. \text{ If } |A| < |B| \text{ and } |C| < |C|_2, \text{ then } |A \times C| < |B \times C|. \]

**Corollary 2.** If \(|A|\) and \(|B|\) are incomparable order types and \(C\) has the fixed point property, then \(C \times A\) and \(C \times B\) are incomparable order types. If \(A\) and \(B\) are incomparable order types and \(|C| < |C|_2\), then \(|A \times C|\) and \(|B \times C|\) are incomparable order types.

**Corollary 3.** If \(|C| < |C|_2\) and \(|A \times C| = |B \times C|\), then \(|A| = |B|\).

Naturally the question arises as to whether or not the conclusion of Theorem 7 remains valid if the hypotheses in (a) and (b) are reversed. The answer is in the negative as is now shown. For (a) let \(|A| = 2, |B| = 1,\) and \(|C| = \omega\). Consider (b). A simply-ordered set \(E\) is said to be complete if every subset \(D\) of \(E\) has both a greatest lower bound and a least upper bound, in \(E\). It is known that a complete set has the fixed point property [6]. Therefore the set \(C\), where \(|C| = 1 + \lambda + 1\), has the fixed point property. Let \(|A| = \omega\) and \(|B| = \omega^*\). Clearly there is no similarity transformation of \(A\) into \(B\). Yet \(|A \times C| = (1 + \lambda + 1)\omega \equiv \lambda \equiv (1 + \lambda + 1)\omega^* = |B \times C|\).

Using the methods of this paper one can show

**Theorem 8.** If \(A\) and \(B\) each have the fixed point property, then the ordered sum \(A + B\) and the product \(A \times B\) each have the fixed point property.

**Bibliography**

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**University of Miami**