AN ALTERNATIVE PROOF OF A THEOREM OF BECKENBACH

EDGAR REICH

Beckenbach [1] has shown that the following analogue of Schwarz's lemma holds:

**THEOREM.** If \( f(z) \) is regular for \(|z| < 1\), and

\[
(1) \quad I(p, \theta) = \int_0^p |f(re^{i\theta})| \, dr \leq 1, \quad 0 \leq p < 1, \quad 0 \leq \theta < 2\pi,
\]

then \( I(p, \theta) \leq p \). Equality for any \((p_0, \theta_0)\), with \( p_0 > 0 \), implies \( f(z) = e^{i\alpha} \), \( \alpha \) real.

We shall derive this theorem by using a device employed by Landau [2] for proving a theorem of Hardy.

By a change of variable of integration, (1) becomes

\[
(2) \quad I(p, \theta) = \rho \int_0^1 |f(\rho e^{i\theta})| \, dt \leq 1, \quad 0 \leq \rho < 1, \quad 0 \leq \theta < 2\pi.
\]

Consider the function

\[
F(p, \theta, z) = z \int_0^1 f(\rho e^{i\theta}) \exp \left[-i \arg (\rho e^{i\theta}) \right] d\rho,
\]

\[0 \leq \rho < 1, \quad 0 \leq \theta < 2\pi, \quad 0 \leq |z| < 1.\]

For every fixed \( \rho \) and \( \theta \), \( 0 \leq \rho < 1, \quad 0 \leq \theta < 2\pi \), \( F \) is an analytic function of \( z \) for \(|z| < 1\), vanishing for \( z = 0 \). Let \( \sigma = \theta + \arg z \). Then

\[
|F(p, \theta, z)| \leq |z| \int_0^1 |f(\rho e^{i\theta})| \, d\rho = |z| \int_0^1 |f(\rho e^{i\sigma})| \, d\rho \leq 1
\]

by (2). Schwarz's lemma now implies that

\[
|F(p, \theta, z)| \leq |z|.
\]

In particular, for \( z = \rho \), we have

\[
|F(p, \theta, \rho)| = F(p, \theta, \rho) = I(p, \theta) \leq \rho,
\]

Received by the editors October 8, 1953 and, in revised form, January 18, 1954.

The author wishes to thank Professor Beckenbach for suggesting the desirability of finding a purely complex-variable proof of this result which had previously been derived only by subharmonic function methods [1].

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as was to be shown. The discussion for the case of equality in the conclusion of the theorem is straightforward.

More generally, by using a stronger form of Schwarz's lemma one can show that if in the hypothesis of the theorem one adds the statement that \( f(z) \) has a zero of order \( n \) at the origin, then one can actually conclude that \( I(\rho, \theta) \leq \rho^n \).

References


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