NOTE ON FUNCTIONS BOUNDED AND ANALYTIC IN THE UNIT CIRCLE

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W. Gross [1] was the first to give an example of an entire function \( w = \phi(\zeta) \) with the property that, for each complex number \( w \), there exists an uncountable set of asymptotic paths along which \( \phi(\zeta) \) tends to \( w \). We shall consider, in this note, the corresponding problem for functions analytic in the unit circle, \( U: |z| < 1 \). Let \( \phi(\zeta) \) be the entire function constructed by Gross, and let \( L \) be one of the paths in the set of asymptotic paths corresponding to an asymptotic value \( \alpha \). Let us cut the \( \zeta \)-plane along \( L \) and map the slit region conformally onto the unit circle \( U \) by means of the function \( \zeta = \zeta(z) \). The composed function \( \phi(\zeta(z)) \) is analytic in \( |z| < 1 \), and, at the point \( z_0 \) on \( |z| = 1 \) corresponding to \( \zeta = \infty \), has the property that every complex value is an asymptotic value of \( \phi(\zeta(z)) \) on an uncountable set of paths terminating at \( z_0 \). However, if we assume a function defined in \( U \) to be bounded, it has at most one asymptotic value at every point of \( |z| = 1 \) and tends to this asymptotic value radially if this exists. Thus it is natural to ask whether there exists a bounded analytic function in \( |z| < 1 \) for which the set of radial limits at a set of linear measure zero on \( |z| = 1 \) fills an open set.

**Theorem.** There exists a function \( w = f(z) \), bounded and analytic in \( |z| < 1 \), such that \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) has modulus 1 for almost all \( e^{i\theta} \) on \( |z| = 1 \), and such that, for each \( \alpha \) with \( |\alpha| \leq 1 \), the equation \( f(e^{i\theta}) = \alpha \) is satisfied on an uncountable set of \( e^{i\theta} \) on \( |z| = 1 \).

We first divide \( |w| \leq 1 \) into four parts:

\begin{align*}
A(1): 0 \leq \psi \leq \pi/2; \quad A(2): \pi/2 \leq \psi \leq \pi; \quad A(3): \pi \leq \psi \leq 3\pi/2; \\
A(4): 3\pi/2 \leq \psi \leq 2\pi \quad (w = re^{i\psi}),
\end{align*}

and draw two disjoint slits \( s(i) \) and \( s(i') \) in the interior of each \( A(i) \) \( (1 \leq i \leq 4) \). Next we divide \( A(1) \) into four parts:

\begin{align*}
A(1, 1): 0 \leq \rho \leq 1/2, \quad 0 \leq \psi \leq \pi/4; \\
A(1, 2): 1/2 \leq \rho \leq 1, \quad 0 \leq \psi \leq \pi/4; \\
A(1, 3): 0 \leq \rho \leq 1/2, \pi/4 \leq \psi \leq \pi/2;
\end{align*}

Presented to the Society, November 28, 1953; received by the editors October 14, 1953.

1 This work was done at Harvard University under Contract N5ori-07634, NR043-046, with the Office of Naval Research.
and draw two disjoint slits \(s(1, j)\) and \(s(1, j')\), disjoint also from \(s(1)\) and \(s(1')\), in the interior of each \(A(1, j)\) \((1 \leq j \leq 4)\). We do the same thing for \(A(2)\), \(A(3)\), and \(A(4)\). Repeating the division into four parts we obtain sectors \(\{A(i, j, \ldots, k)\}\) and slits \(\{s(i, j, \ldots, k)\}\) and \(\{s(i, j, \ldots, k')\}\). We denote by \(F(i, j, \ldots, k)\) (and by \(F(i, j, \ldots, k')\)) the extended \(w\)-plane slit along the nine slits: \(s(i, j, \ldots, k)\) \((s(i, j, \ldots, k')\) resp.), \(\{s(i, j, \ldots, k, l)\}\), and \(\{s(i, j, \ldots, k, l')\}\) \((l = 1, 2, 3, 4)\). \((F(0)\) is the one slit along \(\{s(i)\}\) and \(\{s(i')\}\) \((i = 1, 2, 3, 4)\).)

We now connect \(F(0)\) with \(F(1), F(1'), F(2), \ldots, F(4')\) along \(s(1), s(1'), s(2), \ldots, s(4')\) so that we obtain branch points at the end points of \(s(i)\) and \(s(i')\) \((1 \leq i \leq 4)\), and denote the surface thus obtained by \(G_1\). Next connect \(G_1\) with \(\{F(i, j)\}\) and \(\{F(i, j')\}\) \((i, j = 1, 2, 3, 4)\) (we take two replicas of each of these \(F)\) along 64 boundary slits of \(G_1\) and denote the resulting surface by \(G_2\). We continue this procedure and obtain a surface \(G_n\) at each step. If we denote the surface \(U_n G_n\) by \(G\), \(\{G_n\}\) forms an exhaustion of \(G\).

We shall show that, if we take the lengths of the slits sufficiently small, \(G\) has a null boundary. We draw a disc in \(F(0)\), say \(|w| \geq 1\), exclude it from \(G_n\), and denote the remaining domain by \(G_n'\). We denote by \(\omega_n(P_0)\) the harmonic measure, at a fixed point \(P_0\) of \(G_n\), of the boundary slits of \(G_n\) with respect to the domain \(G_n'\). If all the boundary slits of \(G_n\) reduce to points, \(G_n\) has obviously a null boundary. Therefore if we choose the lengths of these slits sufficiently small, \(\omega_n(P_0)\) is less than \(1/n\). Suppose that all slits are chosen in this way. Then \(\omega_n(P_0) \to 0\) as \(n \to \infty\). This means that \(G\) has a null boundary since \(\{G_n\}\) is its exhaustion.

It is easily seen that \(G\) is of planar character. We can map \(G\) onto a domain \(D\) in the \(W\)-plane conformally in a one-to-one manner. Then, since \(G\) has a null boundary, \(D\) is bounded by a closed set \(E\) of capacity zero. We cut out that part of \(G\) lying over \(|w| \geq 1\) and denote the remaining connected surface over \(|w| < 1\) by \(R\). This corresponds to a subdomain \(D'\) of \(D\), bounded by \(E\) and a countable number of closed curves \(\{\Gamma_i\}\) each of which is an image of \(|w| = 1\). We map the universal covering surface of \(D'\) onto \(U: |z| < 1\) and denote the corresponding function which maps \(U\) onto \(D'\) by \(W(z)\). Then by [2, p. 198], \(W(z)\) tends to a point of \(E\) radially only on a set of linear measure zero on \(C\). Therefore almost all radial limits of \(W(z)\) lie on \(\{\Gamma_i\}\). Consequently, if we compose the function \(W(z)\) with the transformation \(W \to w\) and if we denote the composed func-
tion by \( f(z) \), then \( f(z) \) has almost everywhere a radial limit of modulus one.

We shall show finally that, for any point \( \alpha \) with \( |\alpha| \leq 1 \), there are points of the power of the continuum on \( C \) at which \( f(z) \) tends to \( \alpha \) radially. We select a nested sequence \( A(i) \supset A(i, j) \supset \cdots \rightarrow \alpha \). We start from the origin of the first sheet of \( R \) (a part of \( F(0) \)), enter into the second sheet of \( R \) by crossing over the slit \( s(i) \) or \( s(i') \), enter into the third sheet through \( s(i, j) \) or \( s(i, j') \), and so on. It is possible to limit the path on the Riemann surface in such a way that, as we move from one slit to another, the \( w \)-projection of the path converges to the point \( \alpha \). Since we may obtain nonhomotopic paths in an uncountable number of ways by choosing \( i \) or \( i' \), \( j \) or \( j' \), etc, we obtain a non-countable set of paths terminating at points of \( |z| = 1 \) such that \( f(z) \rightarrow \alpha \) along any one of these paths. Then by Lindelöf's theorem, \( f(z) \) tends radially to \( \alpha \) along a set of radii of the power of the continuum. Hence the theorem is proved.

Bibliography