A BOUND FOR A DETERMINANT WITH
DOMINANT MAIN DIAGONAL

J. L. BRENNER

In [3, pp. 13-14], Hadamard proved that if a square matrix

\[(a_{ij})_{1 \leq i, j \leq n}\]
satisfies the inequalities

\[a_{ii} \neq 0,\]
\[\sigma_i \left| a_{kk} \right| = \sum_{j \neq i} \left| a_{ij} \right|, \quad 0 \leq \sigma_i < 1 \quad (i = 1, 2, \ldots, n),\]

then the determinant of the matrix is different from 0. Upper and lower bounds have been given for the determinant of the matrix; see [4; 5; 6; 7].1 In this article, improvements are given to the bounds of [5; 6]. It is remarkable that the bound of [5] could be improved by using nothing more complicated than ratios already introduced in that article: the bound for \(|\det A| - |a_{11} \cdots a_{nn}|\) given there was asymptotically of the correct order in the \(l_j\) and \(r_j\). It is possible to improve the bounds of [1a] by the methods here given.


The following lemma is needed.

**Lemma.** Let \(A = (a_{ij})_{1 \leq i, j \leq n}\) be a matrix such that the relations (1) hold. Let \(b_{ij}\) be defined as \(a_{ij} - a_{i1}a_{1j}/a_{11}\). Then \((b_{ij})_{2 \leq i, j \leq n}\) has dominant main diagonal, and indeed the relations

\[\sigma_i \left| b_{ii} \right| \geq \sum_{j > i; j \neq i} \left| b_{ij} \right|, \quad \sigma_i \left| b_{ii} \right| \neq 0\]

hold for \(i = 2, 3, \ldots, n\).

This lemma states that the constant \(\sigma_i^j\) defined by the relation \(\sigma_i^j \left| b_{ii} \right| = \sum_{j > i; j \neq i} \left| b_{ij} \right|\) is no greater than the corresponding constant \(\sigma_i\). The fact that \(\det (b_{ij})\) is not 0 is established in [2].

**Proof.** The asserted inequalities follow from the hypotheses as follows.

Presented to the Society, May 1, 1954; received by the editors December 28, 1953.

References 1a, 1b, 6, 7 and 2 added subsequently. The author wishes to thank the referee for calling his attention to the last.
\[ \sigma_i \mid b_{ii} \mid \geq \sigma_i \mid a_{ii} \mid - \sigma_i \mid a_{11}a_{ii}/a_{11} \mid \]
\[ \geq \sum_{j > 1; j \neq i} \mid a_{ij} \mid + \mid a_{i1}/a_{11} \mid \left( \sum_{j > 1; j \neq i} \mid a_{ij} \mid + \mid a_{i1} \mid (1 - \sigma_i) \right) \]
\[ \geq \sum_{j > 1; j \neq i} \mid a_{ij} \mid + \mid a_{i1}/a_{11} \mid \left( \sum_{j > 1; j \neq i} \mid a_{ij} \mid + \mid a_{i1} \mid (1 - \sigma_i) \right) \]
\[ \geq \sum_{j > 1; j \neq i} \mid b_{ij} \mid . \] Similarly, \[ \mid b_{ii} \mid > \sum_{j > 1; j \neq i} \mid b_{ij} \mid . \]

**Theorem 1.** Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a square matrix satisfying (1). Then a bound for \( \det A \) is
\[ \mid \det A \mid \geq \mid a_{11} \mid \prod_{i > 1} \left( \mid a_{ii} \mid - l_i + L_i \right), \]
where \( l_i = \sum_{j < i} \sigma_j \mid a_{ij} \mid ; L_i = \mid a_{ii}/a_{11} \mid \sum_{j > i} \mid a_{ij} \mid . \) Thus \( \mid a_{ii} \mid - l_i \) is automatically positive, and \( L_i \) is non-negative.

In both [5] and [6], a similar bound appears with \( L_i \) replaced by zero. Leaving this aside, the factor \( a_{ii} - l_i \) is still an improvement over the corresponding factor in [6], where \( l_i \) is replaced by the sum \( \sum_{j < i} \mid a_{ij} \mid , \) and is also an improvement over the corresponding factor in [5], where \( l_i \) is replaced by \( \sigma_i \sum_{j < i} \mid a_{ij} \mid , \) \( \sigma_i \) being the greatest of the \( \sigma_j \) with \( j \) not equal to \( i \). Even in this form, the statement can be improved to read that \( \sigma_i \) is the greatest of the \( \sigma_j \) with \( j \) less than \( i \).

This theorem is a special case of

**Theorem 2.** Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a matrix such that relations (1) hold. A bound for \( \mid \det A \mid \) is given by the relation
\[ (2) \mid \det A \mid \geq \prod_{i < k} \left( \mid a_{jj} \mid - r_j + R_j \right) \cdot a_{kk} \cdot \prod_{i > k} \left( \mid a_{jj} \mid - l_j + L_j \right), \]
where \( r_j, l_j, R_j, L_j \) are defined by the relations
\[ r_j = \sum_{t > j} \sigma_t \mid a_{jt} \mid , \quad L_j = \mid a_{jk}/a_{kk} \mid \left( \sum_{t > j} \mid a_{kt} \mid + \sum_{t < k} \mid a_{kt} \mid \right), \]
\[ R_j = \mid a_{jk}/a_{kk} \mid \sum_{t < j} \mid a_{kt} \mid , \quad l_j = \sum_{k \geq t < j} \sigma_t \mid a_{jt} \mid , \]
so that \( R_j \) and \( L_j \) are non-negative.

To prove Theorem 2, first reduce to 0 the nondiagonal elements of the \( k \)th row of \( A \) by adding an appropriate multiple of the \( k \)th column to the other columns. In the resulting matrix, call \( B \) the submatrix obtained by leaving out the \( k \)th row and the \( k \)th column. The \((i, j)\) element \( b_{ij} \) of \( B \) is \( a_{ij} - a_k a_{ik}/a_{kk} \) \((i \neq k, j \neq k)\). As an induc-
tion hypothesis, it is assumed that det $B$ satisfies the relation
\[
|\det B| \geq \prod_{j<k} \left( |b_{jj}| - r'_j + R'_j \right) \cdot |b_{k+1,k+1}|
\]
(3)
\[
\cdot \prod_{j>k+1} \left( |b_{jj}| - l'_j + L'_j \right),
\]
where $R'_j$, $L'_j$ are non-negative, and $r'_j$, $l'_j$ are defined by the relations
\[
r'_j = \sum_{i>j; t=k} \sigma_t |b_{jt}|,
\]
\[
l'_j = \sum_{k<i<j} \sigma_t |b_{jt}|.
\]
The theorem follows from a set of three estimates, which are established below by applying Lemma 1. The first estimate is that provided by the relations
\[
|b_{k+1,k+1}| \geq |a_{k+1,k+1}| - |a_{k+1,k}| \left( \sigma_k |a_{kk}| - \sum_{t \neq k, k+1} |a_{kt}| \right)
\]
\[
= |a_{k+1,k+1}| - \sigma_k |a_{k+1,k}| + |a_{k+1,k}/a_{kk}| \sum_{t \neq k, k+1} |a_{kt}|.
\]
The second estimate concerns the factor $|b_{jj}| - l'_j$, when $j$ exceeds $k+1$:
\[
|b_{jj}| - l'_j \geq |a_{jj}| - |a_{jk}/a_{kk}| \left( \sigma_k |a_{kk}| - \sum_{t \neq j, k} |a_{kt}| \right)
\]
\[
- \sum_{k<i<j} \sigma_t |b_{jt}|
\]
\[
\geq |a_{jj}| - \sum_{k<i<j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}|
\]
\[
- |a_{jk}/a_{kk}| \sum_{k<i<j} \sigma_t |a_{kt}|
\]
\[
\geq |a_{jj}| - \sum_{k<i<j} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \left( \sum_{t>j} + \sum_{i<k} \right) |a_{kt}|.
\]
Last, it is necessary to estimate the factor $|b_{jj}| - r'_j$ if $j$ is less than $k$:
\[
|b_{jj}| - r'_j \geq |a_{jj}| - \sigma_k |a_{jk}| + |a_{jk}/a_{kk}| \sum_{t \neq j, k} |a_{kt}| - \sum_{j<i; t \neq k} \sigma_t |b_{jt}|
\]
\[
\geq |a_{jj}| - \sum_{j<i} \sigma_t |a_{jt}|
\]
\[
+ |a_{jk}/a_{kk}| \left( \sum_{t \neq j, k} |a_{kt}| - \sum_{j<i; t \neq k} \sigma_t |a_{kt}| \right)
\]
\[
\geq |a_{jj}| - \sum_{j<i} \sigma_t |a_{jt}| + |a_{jk}/a_{kk}| \sum_{i<j} |a_{kt}|.
\]
It will be observed that the above inequalities neglect certain terms which if retained would lead to an inequality slightly stronger than that of Theorem 2.

**Theorem 3.** Upper bounds for $|\det A|$ are the expressions obtained by mechanically reversing the four signs $-, +, -, +$, which appear on the right side of (2).

**References**


**State College of Washington**