SOME IMPLICATIONS OF SEMI-1-CONNECTEDNESS

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A topological space $X$ is said to be semi-$n$-connected at a point $x \in X$ provided there exists an open set $U$ containing $x$ such that every Čech $n$-cycle on a compact subset of $U$ bounds on $X$. (If $X$ is regular, “a compact subset of” may be omitted.) If $X$ is semi-$n$-connected at every point, then $X$ is said to be semi-$n$-connected [1]. As shown in [1], if a space has a finite 1-dimensional Betti number, it is semi-1-connected, but the converse is not true. The purpose of this article is to prove certain “point set” properties implied by the hypothesis that the space is semi-1-connected.

In what follows it is assumed that $X$ is a separable metric space. The coefficient group will be a field $F$.

**Theorem 1.** If a continuum $X$ is semi-1-connected at $x \in X$, then there exists an open set $V$ containing $x$ such that, if $Q$ is a region (connected open set) with $\overline{Q} \subset V$, then no continuum in $V - Q$ intersects two components of $F(Q)$. ($F(Q)$ shall mean $\overline{Q} - Q$ for any open set $Q$.)

**Proof.** Let $V$ be an open set such that $x \in V$ and every Čech 1-cycle on $V$ bounds on $X$. Suppose $Q$ is a region with $\overline{Q} \subset V$ and that $K$ is a continuum in $V - Q$ which intersects two components $H_1$ and $H_2$ of $F(Q)$. Let $x \in H_1 \cdot K$ and $y \in H_2 \cdot K$ and let $U$ be an open set such that $K + \overline{Q} \subset U \subset \overline{U} \subset V$. The set $\{x\} + \{y\}$ generates a nonzero element $w$ of the reduced zero dimensional group, $\tilde{H}_0(F(Q))$, since $x$ and $y$ belong to different components of $F(Q)$.

Consider the portion of the Mayer-Vietoris sequence for the triad $(\overline{U}; \overline{U} - Q, \overline{Q})$:

$$\xymatrix{ \tilde{f} : H_1(\overline{U}) \ar[r] & \tilde{H}_0(F(Q)) \ar[r]^h & \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(\overline{U} - Q). }$$

The image of $w$ under $h$ is obtained as follows: if $i_1: F(Q) \to Q$ and $i_2: F(Q) \to \overline{U} - Q$ are inclusion mappings, then $h(w) = (i_{1*}(w), i_{2*}(w)) \in \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(\overline{U} - Q)$. Since $\overline{Q}$ is connected, $i_{1*}(w) = 0$, and since both $x$ and $y$ are in the same component of $\overline{U} - Q$, $i_{2*}(w) = 0$. Therefore, $w$...
is in the kernel of \( h \) and, by exactness, there exists a \( z \in H_1(U) \) such that \( g(z) = w \).

Consider now the Mayer-Vietoris sequence for the triad \( (X; X - Q, \overline{Q}) \). If \( j: (U; U - Q, \overline{Q}) \rightarrow (X; X - Q, \overline{Q}) \) is inclusion, then \( j \) induces a homomorphism of the Mayer-Vietoris sequence of \( (U; U - Q, \overline{Q}) \) into that of \( (X; X - Q, \overline{Q}) \). By the definition of such a homomorphism, we have commutativity in the diagram below.

\[
\begin{array}{c}
\cdots \rightarrow H_1(X) \xrightarrow{g'} H_0(F(Q)) \rightarrow \tilde{H}_0(\overline{Q}) \oplus \tilde{H}_0(X - Q) \\
\downarrow j_* \quad (j|F(Q))_* \downarrow \\
\cdots \rightarrow H_1(U) \xrightarrow{g} \tilde{H}_0(F(Q)) \rightarrow \cdots
\end{array}
\]

Since every Čech 1-cycle on \( U \) bounds on \( X \), \( j_*(z) = 0 \); hence, \( g'j_*(z) = 0 \). But \( g'j_*(z) = (j|F(Q))_*g(z) = w \), since \( (j|F(Q))_* \) is the identity, and \( w \neq 0 \). This contradiction implies the theorem is true.

A space \( X \) is said to be locally peripherally connected at \( x \in X \) provided that for every open set \( P \) containing \( x \) there exists an open set \( M \) containing \( x \) and contained in \( P \) such that \( F(M) \) is connected. A point \( x \in X \) is a local separating point if \( V - \{x\} \) is not connected for some open connected set \( V \) containing \( x \). The next theorem proves a relation between these concepts.

**Theorem 2.** If \( X \) is a locally connected continuum which is semi-1-connected at a nonlocal separating point \( x \in X \), then \( X \) is locally peripherally connected at \( x \).

**Proof.** Let \( U \) be an open set such that \( x \in U \) and every Čech 1-cycle on \( U \) bounds on \( X \). If \( X \) is not locally peripherally connected at \( x \), then there exists an open set \( M \) containing \( x \) and contained in \( U \) such that no open set contained in \( M \) and containing \( x \) has a connected boundary. \( M \) may be taken to have property S [2, p. 22, §15.43], to be connected, and to be such that \( M \subseteq U \). For each positive integer \( i \), let \( R_i \) be a region such that \( x \in R_i \), diam \( (R_i) \leq (1/(i+1))\rho(x, F(M)) \), and \( R_i \supseteq \overline{R}_{i+1} \).

Let \( y \in M - R_i \). That every arc \( xy \) in \( U \) has its last point, in the order \( x \) to \( y \), in \( R_i \), in the same component \( H_i \) of \( F(R_i) \), for each \( i \), is implied by Theorem 1. Hence, for each \( i \), \( H_i \) separates \( x \) from \( y \) in \( U \) and, therefore, \( H_i \) separates \( x \) from \( y \) in \( M \). Let \( X_i \) be the component of \( M - H_i \) containing \( x \). Since \( H_{i+1} \subseteq R_i \subseteq X_i \), for each \( i \), it is true that \( X_{i+1} \subseteq X_i \) (see [2, p. 42]). The collection \( \{X_i\} \) then forms a decreasing sequence: \( X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots \).

Suppose that, for some \( i \), \( X_i \subseteq M \). Then, since \( X_i \) is open and since \( F(X_i) = H_i \), which is connected, the selection of \( M \) is contradicted.
Hence, for each $i$, $X_i \cdot F(M) \neq 0$. Select now a sequence of points $\{x_i\}$ such that $x_i \in X_i$, as follows: (1) Choose $x_1$ from $X_1 \cdot F(M)$. (2) If $x_1 \in X_2$, then let $x_2 = x_1$; otherwise, let $x_2$ be any point in $X_2 \cdot F(M)$. (3) Suppose $x_i$ has been chosen, choose $x_{i+1} = x_i$, if $x_i \in X_{i+1}$, or choose $x_{i+1} \in X_{i+1} \cdot F(M)$, if $x_i \notin X_{i+1}$. Suppose that the sequence resulting is infinite. Let $\bar{x}$ be a limit point of $\{x_i\}$, and let $\{x_i\}$ be a subsequence such that if $p < q$, then $i_p < i_q$ and $x_{i_p} \neq x_{i_q}$ and such that $x_{i_n} \to \bar{x}$ as $n \to \infty$. Since $F(M)$ is compact, $\bar{x} \in F(M)$, and, since $M$ is locally connected [2, p. 20, §15.3], there exists a connected set $N$, open in $M$, containing $\bar{x}$ and not intersecting $R_1$. Let $x_{i_p}$ and $x_{i_q}$, where $p < q$, be two points of $\{x_{i_n}\} \cdot N$. $N \subseteq M - H_{i_q}$ and $x_{i_q} \in N$; therefore, $N \subseteq X_{i_q} \subseteq X_{i_{q-1}} \subseteq \cdots \subseteq X_{i_{p+1}} \subseteq X_{i_p}$. Since $x_{i_p} \in N \subseteq X_{i_{p+1}}$, it is true that $x_{i_{p+1}} = x_{i_p}$, and since $x_{i_{p+1}} \in N \subseteq X_{i_{p+2}}$, $x_{i_{p+2}} = x_{i_{p+1}}$, etc. Thus, $x_{i_n} = x_{i_p}$, which is a contradiction.

Therefore all but a finite number of elements of the sequence $\{x_i\}$ are equal. Let $I$ be a positive integer such that $i \geq I$ and $j \geq I$ imply $x_i = x_j$. Call this point $\bar{x}$. Since $x$ is not a local separating point, $M - \{x\}$ is connected. Let, then, $\bar{x}y$ be an arc joining $\bar{x}$ to $y$ in $M - \{x\}$. There exists a positive integer $J$ such that $i \geq J$ implies $\bar{x}_i \cdot F(R_{i_0})$, in the order $\bar{x}$ to $x$ (then $z_1 \in H_{i_0}$, obviously) and let $z_2$ be the last point of $xy \cdot F(R_{i_0})$, in the order $x$ to $y$. Then $z_2 \in H_{i_0}$. The continuum $z_1 \bar{x} + \bar{x}y + yz_2 \subseteq M - R_{i_0} \subseteq U - R_{i_0}$ and intersects two different components of $F(R_{i_0})$, which contradicts the preceding theorem.

Therefore, $X$ is locally peripherally connected at $x$.

**Theorem 3.** If $X$ is a locally connected continuum which is semi-$1$-connected at a nonlocal separating point $x \in X$, then there exist arbitrarily small open sets $V$ containing $x$ such that, if $Q$ is any region contained in $V$, then there exists a component $C$ of $F(Q)$ such that the component of $X - C$ containing $Q$ is contained in $V$.

**Proof.** Let $U$ be an open set containing $x$ and such that every Čech $1$-cycle on $U$ bounds on $X$. We may take $U$ to be connected. By Theorem 2, there exist arbitrarily small open sets containing $x$ and having connected boundaries. Let $V$ be any one of these such that $V \subseteq U$. Suppose $Q$ is any open connected set contained in $V$. Let $pv$ be any arc in $U$ joining $p \in Q$ to $v \in F(V)$. Let $C$ be the component of $F(Q)$ such that the last point $z$ of $F(Q) \cdot pv$, in the order $p$ to $v$, lies in $C$. Theorem 1 implies that every arc joining $p$ to $v$ has its last point in $F(Q)$, in the order $p$ to $v$, in $C$. Since $Q$ is arcwise connected and $F(V)$ is connected, the same statement may be made for
any arc in $U$ joining a point of $Q$ to a point of $F(V)$.

Let $Z$ be the component of $X - C$ containing $Q$. If $Z$ is not contained in $V$, then $Z \cdot F(V) \neq 0$ and, $Z$ being arcwise connected, there exists an arc $pv$ in $Z$, connecting a point $p \in Q$ to $v \in Z \cdot F(V)$. Therefore, $pv \cdot C = 0$. Let $v'$ be the first point, in the order $p$ to $v$, of $pv \cdot F(V)$. Then $pv' \subset V \cup U$ and $pv' \cdot C \neq 0$. This contradiction implies $Z \subset V$.

**Theorem 4.** If $T$ is a compact totally disconnected set in a locally connected continuum $X$ such that no point of $T$ is a local separating point of $X$ and $X$ is semi-1-connected at each $x \in T$, and if $\epsilon$ is a positive number, then there exists a finite open covering $\{Y_1, Y_2, \ldots, Y_n\}$ of $T$ such that (1) $\text{diam} (Y_i) < \epsilon$, for each $i$, (2) $F(Y_i)$ is connected, for each $i$, and (3) $Y_i \cdot Y_j = 0$, if $i \neq j$.

**Proof.** Let $\{U_x \mid x \in T\}$ be an open covering of $T$ such that for each $x \in T$, every Čech 1-cycle on $U_x$ bounds on $X$. By Theorem 2, there exists, for each $x \in T$, an open set $V_x$ such that (1) $V_x \subset U_x$, (2) $F(V_x)$ is connected, and (3) $\text{diam} (V_x) < \epsilon$. Let $\{V_1, V_2, \ldots, V_n\}$ be a finite subcover of the covering $\{V_x \mid x \in T\}$. Since $T$ is compact, there exists a positive number $d$ such that every subset $Q$ of $X$ with $\text{diam} (Q) < d$ and $Q \cdot T \neq 0$ is contained in $V_j$, for some $j \in \{1, \ldots, n\}$. Cover $T$ with connected open sets $W_1, W_2, \ldots$ such that $\text{diam} (W_i) < d$, for each $i$, and $W_i \cdot W_j = 0$, if $i \neq j$. Then, given any $W_i$, there exists a $j \in \{1, \ldots, n\}$ such that $W_i \subset V_j$. Let $\{W_1, \ldots, W_p\}$ be a finite subcover (renumbered). Further, let $x_1, \ldots, x_n$ be points of $T$ such that $U_{x_i} \supset V_i$, for each $i \in \{1, \ldots, n\}$, and, for each $i \in \{1, \ldots, p\}$, let $k_i$ be the smallest integer in $\{1, \ldots, n\}$ such that $V_i \supset W_i$. Choose the component $C_i$ of $F(W_i)$ as in the proof of Theorem 3 using a point $v_{k_i} \in F(V_{k_i})$ so that the component $Z_i$ of $X - C_i$ containing $W_i$ is contained in $V_{k_i}$. (Actually, $Z_i \subset V_{k_i}$, here.) There exist at most $p$ $Z_i$'s.

(i) If $i \neq j$, then either $Z_i \cdot Z_j = 0$, or $Z_i \supset Z_j$, or $Z_j \supset Z_i$.

**Proof.** Suppose $Z_i \cdot Z_j \neq 0$. If $Z_i \cdot C_j = 0$, then $Z_i \subset Z_j$, and if $Z_j \cdot C_i = 0$, $Z_j \subset Z_i$. If neither is true, then both $Z_i \cdot C_j$ and $Z_j \cdot C_i$ are not empty. Since $W_i \subset X - C_j$ and has a boundary point in $Z_j$, which is open, $Z_j \cdot W_i \neq 0$ and $W_i \subset Z_j$. Similarly, $W_j \subset Z_i$. Let $p \in W_i$ and $q \in W_j$. Let $pq$ be an arc in $Z_i$ and let $y$ be the last point of $(pq) \cdot F(W_i)$, in the order $p$ to $q$. Then $y \in Z_i$, hence $y \in C_i$. Let $C'_i$ be the component of $F(W_i)$ containing $y$. Let $pv_{k_i}$ be an arc in $U_{x_{k_i}}$ and let $x$ be the last point of $(pv_{k_i}) \cdot F(W_i)$, in the order $p$ to $v_{k_i}$. Then $x \in C_i$.

Let $qv_{k_i}$ be an arc in $U_{x_{k_i}}$ and let the last point of $qv_{k_i}$ in $F(W_i)$, in the order $q$ to $v_{k_i}$, be $z$. If $(v_{k_i}, z) \cdot W_i = 0$, then $K = xv_{k_i} + v_{k_i}z + W_j + qy$ is a continuum in $U_{x_{k_i}}$ containing points in two different components.
$C_i$ and $C_i'$ of $F(W_i)$, contradicting Theorem 1. If $(v_k,z) \cdot W_i \neq 0$, let $w$ be the last point of $(v_k,z)$ in $F(W_i)$, in the order $v_k$ to $z$. If $w \in C_i'$, then $wz + \overline{W}_j + qy$ provides the same contradiction as above.

If $w \in C_i'$, then consider two cases:

Case 1, $k_i = k_j$. In this case observe that $z \in C_i$ and, since both $p$ and $q$ belong to $Z_j$, there exists an arc $(pq)'$ in $Z_j$. Let $t$ be the last point of $(pq)'$ in $F(W_j)$, in the order $q$ to $p$. Let $C_i'$ be the component of $F(W_j)$ containing $t$. Since $t \in Z_j$ and $Z_j \cap C_j = \emptyset$, $C_i' \neq C_j$. Then $zw \cup U_{x_{ki}} - W_j$, $tp \subset U_{x_{ki}} - W_j$, and $\overline{W}_i \cup U_{x_{ki}} - W_j$. Therefore, $zw + \overline{W}_j + tp$ is a continuum in $U_{x_{ki}} - W_j$ containing points in two components $C_j$ and $C_j'$ of $F(W_j)$, contradicting Theorem 1.

Case 2, $k_i \neq k_j$ (suppose $k_i < k_j$). In this case, we have $W_j \subset V_{k_i}$ but $\overline{W}_j \not\subset V_{k_i}$. For, if it were, then in the selection of $C_j$ the point $v_{ki}$ would have been used; that is, then $k_i = k_j$. Hence $\overline{W}_j \cdot F(V_{k_i}) \neq 0$. Let $x_{x_{ki}}$ and $pq$ be as above. Then $x_{x_{ki}} \subset U_{x_{ki}} - W_i$, $F(V_{k_i}) \subset U_{x_{ki}} - W_i$, and $\overline{W}_j \cup U_{x_{ki}} - W_i$. The continuum $x_{x_{ki}} + F(V_{k_i}) + \overline{W}_j + qy$ is contained in $U_{x_{ki}} - W_i$ and intersects two different components, $C_i$ and $C_i'$, of $F(W_j)$, contradicting Theorem 1.

Therefore, if $i \neq j$, then either $Z_i \cap Z_j = \emptyset$, or $Z_i \subset Z_j$, or $Z_j \subset Z_i$.

Now let $Y_1$ be the sum of all $Z_i$ such that $Z_i \cdot Z_j = 0$. Let $k_2$ be the smallest integer such that $Z_{k_2} \cdot Y_1 = 0$. Let $Y_2$ be the sum of all $Z_i$ such that $Z_i \cdot Z_{k_2} = 0$. Continuing in this way, if $Y_i$ has been defined, let $k_{i+1}$ be the smallest integer such that $Z_{k_{i+1}} \cdot (Y_1 + Y_2 + \cdots + Y_i) = 0$, and let $Y_{i+1}$ be the sum of all $Z_i$ such that $Z_i \cdot Z_{k_{i+1}} = 0$. Let $\{Y_1, Y_2, \ldots, Y_i\}$ be the collection thus obtained. Each $Y_i$, $i \in \{1, \cdots, l\}$, is a connected open set.

(ii) If $i \neq j$, then $Y_i \cap Y_j = \emptyset$.

Proof. Suppose $i < j$. If $Y_i \cdot Y_j = 0$, let $p \in Y_i \cdot Y_j$. Then there exists a $Z_k$ such that $p \in Z_k$ and $Z_k \cdot Z_{k_i} = 0$, and a $Z_m$ such that $p \in Z_m$ and $Z_m \cdot Z_{k_i} = 0$. Therefore $Z_k \cdot Z_m = 0$ and either (1) $Z_k \subset Z_m$ or (2) $Z_m \subset Z_k$. In case (1), $Z_m \cdot Z_{k_i} = 0$, so $Z_m \subset Y_i \cup \sum_{a=1}^{a=j-1} Y_a$. But then $Z_{k_i} \cdot \sum_{a=1}^{a=j-1} Y_a = 0$, which is a contradiction. In case (2), $Z_m \subset Y_i$ also, since $Z_m \subset Z_k$ and $Z_k \subset Y_i$, giving the same contradiction.

(iii) For each $i$, there exists a $k$ such that $Y_i = Z_k$ and a $j$ such that $Y_i \subset V_j$.

Proof. Observe that there is a $k$ such that $Z_k = Y_i$, for each $i$, for if $Z_{k_i} \neq Y_i$, then there is a point $p \in Y_i - Z_{k_i}$, and there is a $j_1$ such that $Z_{j_1} \cdot Z_{k_i} = 0$ and $p \in Z_{j_1}$. Therefore $Z_{j_1} \supset Z_{k_i}$ properly. If $Z_{j_i} \neq Y_i$, then there is a $j_2$ such that $Z_{j_2} \supset Z_{j_i}$ properly. But this process must stop since there are only a finite number of the $Z_i$'s. Hence, there is a $k$ such that $Z_k = Y_i$, for each $i$. That there is a $j$ such that $Y_i \subset V_j$ is obvious.
The collection \( \{ Y_1, \ldots , Y_l \} \) satisfies the conclusion of the theorem, since

1. For each \( i \), there exists a \( j \) such that \( Y_i \subseteq V_j \), so that \( \text{diam} \ (Y_i) \leq \text{diam} \ (V_j) < \varepsilon \).

2. For each \( i \), there exists a \( k \) such that \( Y_i = Z_k \), so that \( F(Y_i) = F(Z_k) = C_k \) is connected.

3. \( Y_i \cdot Y_j = 0 \), if \( i \neq j \).

**Bibliography**


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