ON THE PROOF OF THE MAIN TAUBERIAN THEOREM FOR THE $C_k$- AND $H_k$-METHODS$^1$

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There are many proofs of the now classical one-sided Tauberian theorem for the $C_k$-method:

**Theorem I.** If the $a_\nu$ are real, if $k > 0$ and$^2$

\[
\sum a_\nu = s (C_k),
\]

\[
\nu a_\nu \leq M \quad (\nu = 0, 1, 2, \ldots)
\]

$M$ being independent of $\nu$, then

\[
\sum a_\nu = s.
\]

The proofs are, for general $k$, not quite easy. The following inductive proof, taking the theorem for granted for $k = 1$—this is the fundamental Hardy-Landau theorem$^3$ from 1910—seems to me of simpler structure than all of them, its main difference from these lying in the fact that it uses the series-to-series instead of the sequence-to-sequence transform.

We prove it first for the $H_k$-method, both theorems being identical for $k = 1$:

**Theorem II.** If the $a_\nu$ are real, if $k$ is an integer $\geq 2$, and if

\[
\sum a_\nu = s (H_k)
\]

and (2) are valid, then also (3).

For if we put $a_\nu^{(0)} = a_\nu$ ($\nu = 0, 1, \ldots$), and $a_\nu^{(p+1)} = a_0$ for $p = 0, 1, \ldots$, and

\[
a_\nu^{(p+1)} = a_1^{(p)} + 2a_2^{(p)} + \cdots + \nu a_\nu^{(p)}
\]

\[
\nu(\nu + 1)
\]

then (1') is the same as

\[
\sum a_\nu^{(p+1)} = s \quad \text{for } p = k - 1
\]

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$^2$ The sums $\sum$ are always taken from 0 to $\infty$, if not otherwise stated.

$^3$ For further references see G. H. Hardy, *Divergent series*, Oxford, 1949, notes on §6.1, p. 145.
and the same as

\[ \sum a^{(p)} = s \ (C_1 \text{ or } H_1) \quad \text{for } p = k - 1. \]

But from (2) and (4) it evidently follows that \( \nu a^{(p)} \leq M \) for all \( \nu \) and \( p = 1, 2, \ldots \). Therefore, the theorem taken for granted for \( k = 1 \), from (6) it follows that also

\[ \sum a^{(p)} = s \]

is true for \( p = k - 1 \). Repeating the same argument we find that (7) is true also for \( p = k - 2, \ldots, 1, 0 \) and Theorem II is proved.

To deduce Theorem I from it, we have to use two well known results of the general theory. First, the fact that \( 0 \leq k < l \) and \( \sum a_r = s \ (C_k) \) imply \( \sum a_r = s \ (C_l) \). On account of this we may, in Theorem I too, suppose \( k \) integral and \( \geq 2 \), replacing it by \( l = \lfloor k \rfloor + 1 \) otherwise. And secondly the fact that \( C_k \) implies \( H_k \)—the easier half of the theorem of equivalence of the two methods (for integral \( k \)).

If we attack Theorem I directly the corresponding proof runs as follows: (1) with an integral \( k \geq 2 \) is the same as

\[ \sum b^{(p+1)} = s \quad \text{for } p = k - 1 \]

with

\[ b^{(p+1)} = \frac{1}{n(n + p + 1)} \sum_{\nu=0}^{n} \binom{n - \nu + p}{n - \nu} \nu a_{\nu}. \]

\((n \geq 1, b^{(p+1)} = a_0, b^{(0)} = a_r)\)—this being the \( C_{p+1} \)-series-to-series transform of \( \sum a_r \). Instead of using now one part of the equivalence theorem of the \( C- \) and \( H \)-methods we have to use a similar result of the general theory, namely that \( C_k \) implies \( C_1 C_{k-1} \): If (8) is true for \( p = k - 1 \) we have for the same \( p \)

\[ \sum b^{(p)} = s \ (C_1). \]

Now from (2) evidently it follows again that \( \nu b^{(p)} \leq M \) for all \( \nu \) and \( p = 1, 2, \ldots \). Therefore, the theorem being true for the \( C_1 \)-method, from (9) it follows that also

\[ \sum b^{(p)} = s \]

\[ \text{The } C_1 \text{-series-to-series transform of the series } \sum c_r \text{ is the series } c_0 + \sum_{r=1}^{\infty} \frac{c_r + 2c_{r+1} + \cdots + r c_r}{(r+1)}. \]
is true for \( p = k - 1 \). Repeating, as before, the same argument the theorem follows.

Dr. Jurkat kindly pointed out to me that nearly the same argument yields the following more general

**Theorem III.** Let

\[
(B) \quad \alpha_n = \sum b_n a_n \quad (n = 0, 1, 2, \ldots)
\]

be a series-to-series transform of the series \( \sum a_n \) into the series \( \sum \alpha_n \). Let

\[
(11) \quad a_n = O(c_n)
\]

be a Tauberian condition for the summation method \( B \). Then the same condition (11) is also a Tauberian condition for the method \( B \cdot B = B^2 \) and therefore for all iterated methods \( B^k = B \cdot B^{k-1} \) (\( k = 2, 3, \ldots \)) if

\[
\sum_{\nu} |b_n c_\nu| = O(c_n).
\]

Furthermore, if \( B \) is regular and if (11) is a best Tauberian condition for the method \( B \), it is also a best condition for the method \( B^k \).

This follows at once from the inclusion \( B^{k-1} \subseteq B^k \).

Finally, we remark that Theorem III and the last addition to it are evidently also true, the \( a_n \) being real, for a one-sided Tauberian condition \( a_n = O_R(c_n) \) if the \( b_n \) are \( \geq 0 \) and if \( \sum \nu b_n c_\nu = O_R(c_n) \).

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