A NOTE ON LINEAR VECTOR SPACES OF MAPPINGS WITH POSITIVE JACOBIANS

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Let \( \mathcal{F} \) be a family of continuously differentiable mappings \( f \) of an open set \( D \) in the \( xy \)-plane into the \( uv \)-plane satisfying the following conditions:

(a) \( f, g \in \mathcal{F} \Rightarrow \lambda f + \mu g \in \mathcal{F} \) for all real \( \lambda, \mu \),
(b) \( f \in \mathcal{F} \Rightarrow \) the Jacobian of \( f \) is non-negative and zero only if its rank is zero,
(c) \( \mathcal{F} \) contains two mappings \( f_1 = (u_1, v_1), f_2 = (u_2, v_2) \) such that the Jacobian

\[
\frac{\partial (v_1, v_2)}{\partial (x, y)} \neq 0
\]

for \( (x, y) \in D \).

The mappings described by functions \( f = u + iv \) regular analytic in \( z = x + iy \), \( (x, y) \in D \), form such a family. J. E. McLaughlin and C. J. Titus [1] proved the converse: If the two mappings of condition (c) are described by functions regular analytic in \( z = x + iy \), then \( \mathcal{F} \) is a linear vector space of functions regular analytic in \( z = x + iy \), \( (x, y) \in D \). It is the purpose of this note to give a characterization of the general class \( \mathcal{F} \).

Consider continuously differentiable solutions \( u(x, y), v(x, y) \) of the system of partial differential equations

\[
L_1 \equiv u_x - a_{11}v_x - a_{12}v_y = 0, \quad L_2 \equiv u_y + a_{21}v_x + a_{22}v_y = 0
\]

where \( a_{11}, a_{12}, a_{21}, a_{22} \) are real functions of \( x, y \) continuous in \( D \). If the system is elliptic [2], \( (a_{11} + a_{22})^2 < 4a_{12}a_{21} \) and \( a_{12} > 0, a_{21} > 0 \) in \( D \), then the Jacobian

\[
\frac{\partial (u, v)}{\partial (x, y)} = a_{21}v_x + (a_{11} + a_{22})v_xv_y + a_{12}v_y
\]

is positive unless it is of rank 0. Therefore, the mappings \( f \) described by functions \( u, v \) continuously differentiable in \( D \) which are solutions of the elliptic system (1) form a family \( \mathcal{F} \). We prove the converse.

Theorem. The mappings \( f \) of a family \( \mathcal{F} \) are described by functions \( u, v \) that are solutions of a unique elliptic system of form (1).

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To prove this consider the mappings \( f_1, f_2 \) satisfying condition (c). Then we can solve the system

\[
\begin{align*}
 u_{ix} - a_{11} v_{ix} - a_{12} v_{iy} &= 0, & u_{iy} + a_{21} v_{ix} + a_{22} v_{iy} &= 0 \\
 (i = 1, 2)
\end{align*}
\]

for \( a_{11}, a_{12}, a_{21}, a_{22} \), and the solution is uniquely determined. The system

\[
\begin{align*}
 L_1 &= u_x - a_{11} v_x - a_{12} v_y = 0, & L_2 &= u_y + a_{21} v_x + a_{22} v_y = 0
\end{align*}
\]

is then satisfied by the linear combinations \((u, v)\) of \((u_1, v_1), (u_2, v_2)\) with real constant coefficients, and since the corresponding mappings \( f \) are in \( \mathcal{G} \) we have by (b)

\[
\frac{\partial(u, v)}{\partial(x, y)} = a_{21} v_x^2 + (a_{11} + a_{22}) v_x v_y + a_{12} v_y^2 > 0
\]

except where \( v_x = v_y = 0 \). Since every real vector \((X, Y)\) is a linear combination with real coefficients of the linearly independent vectors \((v_1, v_y), (v_2, v_2)\), the form \( a_{21} X^2 + (a_{11} + a_{22}) XY + a_{12} Y^2 \) is positive definite, \((a_{11} + a_{22})^2 < 4a_{12} a_{21}\), and \( a_{12}, a_{21} > 0 \).

Now let \( f = (u, v) \) be any mapping in class \( \mathcal{G} \). By Lemma 2 of [1] we know that

\[
J(f) = \lambda_1 J(f_1) + \lambda_2 J(f_2)
\]

where \( J(f), J(f_1), J(f_2) \) are the Jacobian matrices of \( f, f_1, f_2 \) and \( \lambda_1, \lambda_2 \) are scalars that may depend on \( x, y \). It follows that \( u, v \) satisfy the system

\[
L_1 = 0, \quad L_2 = 0,
\]

and the theorem is proved.

**Remark 1.** If the two mappings of condition (c) are described by functions regular analytic in \( z = x + iy \), then \( L_1 = 0, L_2 = 0 \) are the Cauchy-Riemann equations and we obtain the characterization of the analytic functions given in [1].

**Remark 2.** If the two mappings of condition (c) are described by sigma-monogenic functions [3] \( f = u + iv \), where

\[
\sigma_1(x) u_x = \tau_1(y) v_y, \quad \sigma_2(x) u_y = -\tau_2(y) v_x
\]

with \( \sigma_i > 0, \tau_i > 0 \), then \( \mathcal{G} \) contains only sigma-monogenic functions.

**Remark 3.** If the two mappings of condition (c) are described by functions \( u_i, v_i \) analytic in the real variables \( x, y \), then the coefficients \( a_{ij} \) of the system \( L_1 = 0, L_2 = 0 \) are analytic, and by a well known property of such systems, \( \mathcal{G} \) contains only mappings described by functions analytic in \( x, y \).
References


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