and for each particular $f$ the measure may be so chosen that (moreover)

$$7.13 \quad \int_B \log |f(x)| \, m(x) \geq \log |f(s)|.$$ 

Naturally, if for some reason there is for some $s$ only one measure satisfying 7.12, then 7.13 holds for that measure.

**Bibliography**


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**FAMILIES OF CURVES**

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Amasa Forrester in [1] proved the following theorem of a mixed Euclidean and topological character. If $\phi$ is a continuous map without fixed points on the Euclidean $n$-sphere such that $\phi^2$ is the identity, then the chords $P\phi(P)$ for all points $P$ of the sphere completely fill the interior of this sphere.

The object of this note is to generalize this theorem to a purely topological statement.

First we recall the definition of retract. If $B \subseteq A$ are two spaces, then $B$ is a retract of $A$ if there is $r: A \to B$ which leaves fixed all points of $B$. (If $X$ and $Y$ are spaces the symbol $f: X \to Y$ shall denote a continuous map from $X$ to $Y$.)

Let $I$ denote the unit interval. If $F: B \times I \to A$ and $t \in I$, define $F_t: B \to A$ by $F_t(b) = F(B, t)$ for all $b \in B$.

**Observation.** If $F: B \times I \to A$ and if $B$ is a retract of $A$ by the map $r$ and if $p, q \in I$, then $rF_p$ is homotopic to $rF_q$.

In fact such a homotopy is provided by $G: B \times I \to B$ defined by

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$G(b, t) = rF(b, p + (q - p)t)$. Clearly $G_0 = rF_p$ and $G_1 = rF_q$.

Now let $E^{n+1}$ be the topological $n+1$ dimensional cell and $S^n$ its boundary (an $n$ dimensional topological sphere). Now, for any point $P \in E^{n+1} - S^n$, $S^n$ is a retract of $E^{n+1} - \{P\}$ by the map

$$r: E^{n+1} - \{P\} \to S^n$$

defined by carrying over the central projection of the Euclidean cell by a homeomorphism. This fact and the observation yield:

**Proposition 1.** If $f_i: S^n \to S^n$, $i = 0, 1$, are not homotopic and if $F: S^n \times I \to E^{n+1}$ satisfies $F(P, i) = f_i(P)$, $i = 0, 1$, all $P \in S^n$, then $F$ is onto $E^{n+1}$.

**Proposition 2** (generalization of Forrester's theorem). Let $\phi: S^n \to S^n$ be of period $p \neq 1$. Let $F: S^n \times I \to E^{n+1}$ satisfy (a) $F(P, 0) = P$ and (b) $F(P, 1) = F(\phi(P), 1)$. Then $F$ is onto $E^{n+1}$.

**Proof.** Observe first that it is sufficient to prove this for $p$ prime. For if $p$ were not prime and $q$ is a prime dividing $p$, then the hypothesis of Proposition 2 is satisfied with $\phi$ replaced by $\phi^{p/q}$ and the latter is of prime period. In the following proof therefore $p$ is taken to be prime.

Assume on the contrary that there is a point $Q \in E^{n+1} - F(S^n \times I)$. By (a), $Q \in S^n$ and by a previous remark there is a retraction $r: E^{n+1} - \{Q\} \to S^n$. Regarding $F$ as a map into $E^{n+1} - Q$ we would have $rF_0$ homotopic to $rF_1$. Now $rF_0$ is the identity map of $S^n$ (hence of degree 1) while $rF_1$ has the property that $(rF_1)\phi = rF_1$ on account of condition (b).

To conclude the proof it shall be shown that any map $g: S^n \to S^n$ satisfying $g\phi = g$ has a degree divisible by $p$.

By [2] there is a cycle of the form $c + \phi(c) + \phi^2(c) + \cdots + \phi^{p-1}(c)$ in a generator of $H^n(S^n, J_p)$. Calling this cycle $z$ we have $g(z) = pc = 0$ (mod $p$). Thus the degree of $g$ is divisible by $p$. This concludes the proof of Proposition 2.

Forrester's family of straight lines may be described by

$$F: S^n \times I \to E^{n+1}$$

where $F(P, t)$ is the point $Q$ on the line segment joining $P$ to $\phi(P)$ such that $PQ/P\phi(P) = t/2$.

If the notion of homotopy is translated into the language of a continuous family of curves then Proposition 2 becomes:

**Proposition 2'.** If (1) $\phi: S^n \to S^n$ satisfies the condition stated in Proposition 2 and (2) from each point $P$ of $S^n$ there begins one curve of
$E^{n+1}$ so that the curves beginning at $P$ and $\phi(P)$ have the same terminal point and (3) the parametrization of these curves depend continuously on $P$, then this family of curves fills $E^{n+1}$.

**Proposition 3.** Let $R^n$ refer to $n$ dimensional Euclidean space. If for each direction in $R^n$ there is given in a continuous manner precisely one straight line with that direction, then this family of lines fills $R^n$.

**Proof.** Compactify $R^n$ to $E^n$ by adding two points at infinity for each direction in $R^n$. Then apply Proposition 2 or 2' with $p = 2$.

**Proposition 4.** Let $A$ be a compact subset of $R^n$. A necessary and sufficient condition that $A$ be a convex set with the property that each support plane has precisely one contact point is that there exists a continuous choice function on the set of $n-1$ dimensional planes, meeting $A$, with values in $A$. Moreover any such function is onto $A$.

**Proof.** Let $A$ be a convex subset of $R^n$ with the property that each plane of support has one point of contact. Assign to each cross-section its centroid. By Proposition 2 with $p = 2$ it is easy to show that this function is onto $A$ (compare to p. 13 of [3]).

The proof of sufficiency is left to the reader. In a subsequent paper the intersection properties of families of curves will be considered.

**Bibliography**