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A THEOREM ON HYPERSIMPLE SETS

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Introduction. E. L. Post proved in his paper *Recursively enumerable sets of positive integers and their decision problems*¹ that a creative set cannot be reducible to a hypersimple set by truth tables (pp. 308-310). The present paper is concerned with the question whether a creative set can be Turing reducible to a hypersimple set. It will be shown that the answer to this question is affirmative. In fact, we shall establish the following more general statement: *for every recursively enumerable, but not recursive, set α , a hypersimple set β can be effectively found such that α and β are Turing reducible to each other.*

Preliminaries. A non-negative integer is called a *number*, a collection of numbers is called a *set* and a collection of sets a *class*. A set is *immune*, if it is infinite, but has no infinite recursively enumerable subset. For the definitions of a discrete array and a hypersimple set we refer to our paper *Two notes on recursively enumerable sets*² (p. 497). We shall use the following notations: " $\rho(f)$ " for the range of the function $f(n)$, " E " for the class of all recursive sets, " F " for the class of all recursively enumerable (r.e.) sets, " α t -red β " for α is reducible to β by truth tables, and " α red β " for α is Turing reducible to β . If $a(t_1)$ and $a(t_2)$ are elements of the sequence $\{a(n)\}$, we say that $a(t_2)$ *succeeds* $a(t_1)$ in $\{a(n)\}$ if $t_2 > t_1$.

DEFINITION. Let $a(n)$ be a 1-1 function. The element $a(t)$ of the

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¹ Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284-316.

² Proc. Amer. Math. Soc. vol. 4 (1953) pp. 495-501.

sequence $\{a(n)\}$ is called *minimal* if $a(t) < a(t+k)$ for every $k \geq 1$. The set of all elements in $\{a(n)\}$ which are not minimal is denoted by "hyp $a(n)$."

DEFINITION. The subset σ of α is called *hypersimple relative to α* if σ is r.e., $\alpha - \sigma$ infinite, and σ includes at least one row of every discrete array all of whose elements belong to α .

Let $a(n)$ be a 1-1 recursive function. It is now readily verified that the subset σ of $\rho(a)$ is hypersimple relative to $\rho(a)$ if and only if $a^{-1}(\sigma)$ is hypersimple.

1. The main result.

THEOREM 1. For every set $\alpha \in F - E$ one can effectively find a r.e. set β such that: (1) β is hypersimple, (2) β t -red α , (3) α red β .

PROOF. Let $a(n)$ be a 1-1 recursive function ranging over α , $\zeta = a_f \text{hyp } a(n)$ and $\beta = a_f a^{-1}(\zeta)$. We claim that β satisfies the requirements.

(1) It suffices to show that ζ is hypersimple relative to α . We shall call an element of α minimal if it is minimal relative to $\{a(n)\}$. For every number k we can effectively list the elements in $a(0), \dots, a(k)$ which are succeeded by a smaller element in this finite sequence. Since we can do this for $k=1, 2, \dots$, we conclude that ζ is r.e. Suppose $a(k_0)$ is the smallest element of α , then $a(k_0)$ is minimal; if $a(k_1)$ is the smallest element of $\alpha - \{a(0), \dots, a(k_0)\}$, $a(k_1)$ is also minimal and $a(k_0) < a(k_1)$. Continuing this procedure we see that α contains infinitely many minimal elements, i.e., that $\alpha - \zeta$ is infinite. Let $\{\delta_n\}$ be a discrete array all of whose elements belong to α . Then there exist recursive functions $d(n, x)$ and $e(n)$ such that for every n , $\delta_n = \{d(n, 0), \dots, d(n, e(n))\}$; we may assume without loss of generality that $d(n, x)$ is strictly increasing for every n . We know that α is not recursive. To prove that ζ includes at least one row of $\{\delta_n\}$ it suffices therefore to show that the assumption that ζ includes no row of $\{\delta_n\}$ implies that α is recursive. This can be done in the following manner. Suppose ζ included no row of $\{\delta_n\}$. Then every row of $\{\delta_n\}$ would contain at least one minimal element. Observe that $d(n, 0)$ is a 1-1 recursive function of n such that $d(n, 0) = \min \delta_n$ for every n . Let x be any number. Find the first element in $\{d(n, 0)\}$ which is greater than x , say $d(r, 0)$. Then δ_r contains a minimal element, say $d(r, s)$. Thus $x < d(r, 0) \leq d(r, s) \leq d(r, e(r))$. We can now locate the elements of δ_r in $\{a(n)\}$, i.e., we can effectively find the numbers $n_0, \dots, n_{e(r)}$ such that $d(r, 0) = a(n_0), \dots, d(r, e(r)) = a(n_{e(r)})$. Let $t_x = a_f \max(n_0, \dots, n_{e(r)})$. The element $a(t_x)$ is now

either equal to the minimal element $d(r, s)$ or it succeeds $d(r, s)$ in $\{a(n)\}$. It now follows from the fact that $d(r, s) > x$ that all elements in $\{a(n)\}$ which succeed $a(t_x)$ in $\{a(n)\}$ are greater than x . Hence $x \in \alpha$ if and only if $x \in \{a(0), \dots, a(t_x)\}$. Since t_x is a recursive function of x , we would now possess an effective procedure to test whether x belongs to α . Thus α would be recursive.

(2) Let x be any number. Then $x \in \beta$ if and only if $a(x)$ is not minimal. Let $\alpha_x^* = a_f\{0, \dots, a(x) - 1\} - \{a(0), \dots, a(x - 1)\}$, then $a(x)$ is not minimal if and only if α_x^* contains at least one element of α . There now exist recursive functions $a_{x,n}^*$ and $m(x)$ such that:

- (a) if α_x^* is empty, then $a_{x,0}^* = 0$ and $m(x) = 0$,
- (b) if α_x^* is nonempty, then

$$\alpha_x^* = \{a_{x,0}^*, \dots, a_{x,m(x)}^*\}.$$

Let T_x be the truth table of order $m(x) + 1$ whose last column is $-$, $-$ in case α_x^* is empty and is $+$, $+$, \dots , $+$, $-$ in case α_x^* is nonempty. The decision problem of β is now reducible to that of α by the reduction which associates with every number x the finite sequence $a_{x,0}^*, \dots, a_{x,m(x)}^*$ and the truth table T_x . Thus β t -red α .

(3) Let x be any number. We can effectively find a recursive function h_n (depending on x) such that all elements of α which are greater than x are generated in the sequence $\{a(h_n)\}$. We now raise the following questions about ζ :

$$"a(h_0) \in \zeta?," \quad "a(h_1) \in \zeta?," \dots$$

After a finite number of questions the correct answer must be negative, since $\alpha - \zeta$ is infinite and all but finitely many elements of α are assumed by $a(h_n)$. Let $p(x)$ be the first n such that $a(h_n) \notin \zeta$. Then we know for every x that $a(h_{p(x)})$ is a minimal element of α greater than x . Thus $x \in \alpha$ if and only if $x \in \{a(0), \dots, a(h_{p(x)} - 1)\}$. Let $c_\alpha(x)$, $c_\beta(x)$, $c_\zeta(x)$ be the characteristic functions of α , β , ζ respectively. We have proved that $c_\alpha(x)$ is recursive in $c_\zeta(a(x))$. But $c_\zeta(a(x)) = c_\beta(x)$. It follows that $c_\alpha(x)$ is recursive in $c_\beta(x)$, i.e., that α red β .

REMARK 1. In the proof of (1) no use is made of the fact that there exists a hypersimple set. As a side result we have therefore a new existence proof for hypersimple sets.

REMARK 2. While the fact that $\alpha = \rho(a)$ is not recursive plays an essential role in the proof of (1), it is not used in the proof of (2). The statement β t -red α is therefore still valid if α is recursive. Thus, if α is recursive, β is recursive. This can also be seen in the following way. If $a(n)$ is a 1-1 recursive function, ranging over the recursive set α ,

we can decide for any n whether $a(n)$ is minimal in $\{a(n)\}$ by locating $a(n)$ in the recursive enumeration of α according to size.

COROLLARY 1. *There exist two r.e. sets σ and τ such that σ t -red τ is false, but σ red τ true.*

PROOF. Let σ be any creative set, $s(n)$ a 1-1 recursive function ranging over σ , and $\tau =_{df} s^{-1}(\text{hyp } s(n))$. Then σ red τ is true; but σ t -red τ is false, because a creative set cannot be reducible to a hypersimple set by truth tables.

COROLLARY 2. *There exist two sets in $F-E$ which are not Turing reducible to each other if and only if there exist two hypersimple sets which are not Turing reducible to each other.*

2. A generalization.

DEFINITION. Let $a(n)$ be a 1-1 function and t any number. Then the *deficiency* $\text{dfc } a(t)$ of the element $a(t)$ in $\{a(n)\}$ is the number of elements which while smaller than $a(t)$ succeed $a(t)$ in $\{a(n)\}$.

Let $a(n)$ be a 1-1 recursive function. It is now easily verified that the function $\text{dfc } a(n)$ is recursive if and only if the set $\rho(a)$ is recursive. For let $\rho(a)$ be recursive. Then the statements

$$0 \in \rho(a), 1 \in \rho(a), \dots, a(n) - 1 \in \rho(a)$$

can be effectively tested for every n ; the results can then be compared with the elements $a(0), \dots, a(n-1)$. Thus $\text{dfc } a(n)$ is recursive. To establish the converse, suppose $\rho(a)$ is not recursive. Then the set λ of all elements in $\{a(n)\}$ which have deficiency 0 (i.e., which are minimal) is immune by Theorem 1. But, if $\text{dfc } a(n)$ were recursive, λ would be r.e. We conclude that $\text{dfc } a(n)$ is not recursive.

THEOREM 2. *Let $a(n)$ be a 1-1 recursive function such that $\alpha = \rho(a) \in F-E$. Let k be any number ≥ 1 and $\beta_k =_{df} \{n \mid \text{dfc } a(n) \geq k\}$. Then we know: (1) β_k is hypersimple, (2) β_k t -red α , (3) α red β_k .*

PROOF. Let $\zeta_k =_{df} a(\beta_k)$. It is now easily verified that ζ_k is r.e.

(1) It suffices to show that ζ_k is hypersimple relative to α . This can be done by induction on k . ζ_1 is hypersimple relative to α . The set $\alpha - \zeta_k$ is infinite, since it includes the infinite set $\alpha - \zeta_1$. Suppose ζ_{k^*} is hypersimple relative to α , where $k^* \geq 1$. Assume that ζ_{k^*+1} were not hypersimple relative to α . Then there would exist a discrete array of elements in α , none of whose rows would be included in ζ_{k^*+1} , say $\{\delta_n\}$. Each row of $\{\delta_n\}$ would therefore contain at least one element with a deficiency $\leq k^*$. We claim that $\{\delta_n\}$ must have infinitely

many rows containing at least one element with deficiency k^* ; for otherwise $\{\delta_n\}$ would have a discrete subarray each row of which would contain at least one element with a deficiency $\leq k^* - 1$; this would contradict the induction hypothesis that ζ_{k^*} was hypersimple relative to α . Since $\{\delta_n\}$ has infinitely many rows which contain at least one element with deficiency k^* , we can, by comparing $\{\delta_n\}$ with a recursive enumeration of ζ_{k^*} , obtain a subarray of $\{\delta_n\}$ each row of which contains only elements with a deficiency $\geq k^*$ and at least one element with deficiency k^* . Let $\{\rho_n\}$ be such a subarray of $\{\delta_n\}$. We now transform ρ_0 into σ_0 in the following manner:

(a) replace every element x of ρ_0 by the first element in $\{a(n)\}$ which while smaller than x succeeds x ,

(b) omit all repetitions which might arise because different elements of ρ_0 are replaced by the same element.

Let $t =_{df} \max \{n \mid a(n) \in \sigma_0\}$. Only finitely many rows of $\{\rho_n\}$ can contain at least one of the elements $a(0), \dots, a(t)$; thus we can effectively find the first row of $\{\rho_n\}$ all of whose elements succeed $a(t)$ in $\{a(n)\}$, say ρ_i . We now define σ_1 in terms of ρ_i as we defined σ_0 in terms of ρ_0 . Continuing this procedure we transform the discrete array $\{\rho_n\}$ into a discrete array $\{\sigma_n\}$. It now follows from the properties of $\{\rho_n\}$ and the definition of $\{\sigma_n\}$ that every row of $\{\sigma_n\}$ contains at least one element with a deficiency $\leq k^* - 1$. Thus ζ_{k^*} would include no row of the discrete array $\{\sigma_n\}$ of elements in α ; this would contradict the assumption that ζ_{k^*} is hypersimple relative to α . We conclude that ζ_{k^*+1} is also hypersimple relative to α .

(2) $x \in \beta_k$ if and only if $a(x)$ has a deficiency $\geq k$. We now define α_x^* , $a_{x,n}^*$, and $m(x)$ in the same way as in the proof of the second part of Theorem 1. Let $T_{k,x}$ be the truth table of order $m(x) + 1$ such that: if α_x^* is empty the last column of $T_{k,x}$ is $-$, $-$, if α_x^* is nonempty the last column of $T_{k,x}$ contains the symbol $+$ in the t th row if and only if the truth assignment in the t th row contains the symbol $+$ at least k times. The decision problem of β_k is now reducible to that of α by the reduction which associates with every number x the finite sequence $a_{x,0}^*, \dots, a_{x,m(x)}^*$ and the truth table $T_{k,x}$. Thus β_k t -red α .

(3) We apply again induction on k . We know that α red β_1 . Suppose α red β_{k^*} , where $k^* \geq 1$. The fact that $\zeta_{k^*+1} \subset \zeta_{k^*}$ implies that $\beta_{k^*+1} \subset \beta_{k^*}$. Suppose $\beta_{k^*} - \beta_{k^*+1}$ is finite. Then α red β_{k^*+1} follows from α red β_{k^*} , because β_{k^*} is many-one reducible to β_{k^*+1} . Now suppose $\beta_{k^*} - \beta_{k^*+1}$ is infinite, then $\zeta_{k^*} - \zeta_{k^*+1}$ is also infinite. We can effectively find a 1-1 recursive function c_n such that ζ_{k^*} is generated in the sequence $a(c_0), a(c_1), \dots$. We now raise the following questions about ζ_{k^*+1} :

(I) “ $a(c_0) \in \zeta_{k^*+1}?$ ”, “ $a(c_1) \in \zeta_{k^*+1}?$ ”, \dots

Among the correct answers infinitely many must be negative, since $\zeta_{k^*} - \zeta_{k^*+1}$ is infinite. Let $\{d_n\}$ be the sequence of all m for which there exists an n such that $m = a(c_n) \notin \zeta_{k^*+1}$ ordered in the same way as they occur in (I). Observe that for every n the deficiency of d_n is exactly k^* . Let e_n be the minimum of the k^* elements in $\{a(n)\}$ which while smaller than d_n succeed d_n . Now suppose x is any number. Since there exists an n_x such that $\{a(n)\}$ contains only elements $> x$ after $a(n_x)$, there must be a number $t(x)$ such that $e_{t(x)} > x$. But $e_{t(x)}$ is minimal, for the definition of e_n implies that e_n is minimal for every n . Hence $x \in \alpha$ if and only if $x \in \{a(0), a(1), \dots, e_{t(x)}\}$. The number $e_{t(x)}$ can be obtained after finitely many questions in (I) are correctly answered. Let $f(x), g(x), h(x)$ be the characteristic functions of $\alpha, \beta_{k^*+1}, \zeta_{k^*+1}$, respectively. We have proved that $f(x)$ is recursive in $h(a(x))$. But $h(a(x)) = g(x)$. It follows that $f(x)$ is recursive in $g(x)$, i.e., that α red β_{k^*+1} .

COROLLARY. *Let $a(n)$ be a 1-1 recursive function such that $\rho(a)$ is not recursive. Let k be any number ≥ 1 . Then:*

- (1) *there are infinitely many elements in $\{a(n)\}$ with a deficiency $\geq k$,*
- (2) *the set of all elements in $\{a(n)\}$ with a deficiency $< k$ is immune,*
- (3) *the set of all elements in $\{a(n)\}$ with deficiency k is finite or immune.*

PROOF. The first two parts follow immediately from the fact that ζ_k is hypersimple relative to $\rho(a)$. The third part follows from the second part, because the set of all elements with deficiency k is included in the set of all elements with a deficiency $< k+1$.

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