ON THE MINIMUM MODULUS OF A ROOT OF A POLYNOMIAL

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Landau showed [1] that the equation \( 1 + z + \alpha z^n = 0, \ m > 1 \), has at least one root with the modulus \( \leq 2 \) and that the equation

\[
1 + z + \alpha z^m + \beta z^n = 0 \quad (1 < m < n)
\]

has at least one root with the modulus \( \leq 17/3 \). He posed the problem whether the equation

\[
P_K(z) = 1 + z + \alpha_1 z^{n_1} + \cdots + \alpha_{K-1} z^{n_{K-1}} = 0
\]

has at least one root with a modulus not greater than a number \( M(K) \), depending only on the number of terms \( P_K(z) \) and not at all on the numbers \( \alpha_1, \alpha_2, \cdots, \alpha_{K-1}, n_1, n_2, \cdots, n_{K-1} \).

This problem was solved by P. Montel. Montel [2] in his paper, written in 1923, showed that the number \( M(K) \) has the simple value \( K \), and that whenever the root assumes this maximum value \( K \) all the roots of the polynomial are equal to \(-K\).

In this note we establish the following stronger result.

**Theorem.** The equation \( P_K(z) = 1 + z + \alpha_1 z^{n_1} + \cdots + \alpha_{K-1} z^{n_{K-1}} = 0 \) (2 \( \leq n_1 < n_2 < n_3 < \cdots < n_{K-1} \), \( \alpha_i \neq 0 \) (i = 1, 2, \cdots, K-1) has at least one root within or on the circumference of a circle with the center \(-\lambda/2\) and radius \( \lambda/2 \) where \( \lambda = (n_1/(n_1-1)) \cdot (n_2/(n_2-1)) \cdots (n_{K-1}/(n_{K-1}-1)) \).

**Proof.** To prove the theorem and simplify operations let us consider the equation \( f(z) = 1 + z + \alpha z^m + \beta z^n + \gamma z^p = 0 \) (2 \( \leq m < n < p \)). Putting \( z = 1/\delta \) we have \( \phi(\delta) = \delta^p + \delta^{p-1} + \alpha \delta^{p-m} + \beta \delta^{n-m} + \gamma = 0 \); the derivative may be written \( \phi'(\delta) = \delta^{p-1} \cdot \phi_1(\delta) \) where

\[
\phi_1(\delta) = \beta \cdot \delta^n + \alpha(p-m)\delta^{n-m} + \beta(p-n).
\]

Similarly we write \( \phi_1'(\delta) = \delta^{n-m-1} \cdot \phi_2(\delta) \) where \( \phi_2(\delta) = \beta \cdot n \delta^m + \alpha(p-m)(n-m) \). Similarly we write \( \phi_2'(\delta) = \beta \cdot m n \delta^{m-2} \cdot (\delta + \theta) \) where \( \theta = (p-1)/\delta \cdot (n-1)/n \cdot (m-1)/m \).

Let \( \Pi(x) \) be the half-plane \( R(\delta) \geq x \). If \( 0 > x > -\theta, \phi_2(\delta) \) must have at least one root not in \( \Pi(x) \), since otherwise by Lucas' Theorem [3] all the roots of \( \phi_2'(\delta) \) would be in \( \Pi(x) \), and this contradicts the fact that \( \delta = -\theta \) is a root of \( \phi_2(\delta) \). Similarly \( \phi_1(\delta) \) must have at least one

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1 Numbers in brackets refer to the list of references.
root not in \( \Pi(x) \), and hence \( \phi(\delta) \) must have at least one root not in \( \Pi(x) \). But \( \phi(\delta) \) has only a finite number of roots, so that if all these roots were in \( R(\delta) > -\theta \) one could select an \( x > -\theta \) so that all the roots would be in \( \Pi(x) \). Hence \( \phi(\delta) \) has at least one root in \( R(\delta) \leq -\theta \), and therefore \( f(z) \) has at least one root in the image of this half-plane under \( 1/z \), namely the circle \( |z + 1/2\theta| \leq 1/2\theta \). It is easy to see that in the general case \( 1/\theta \) is replaced by \( \lambda \) and this completes the proof of the theorem.

Since \( \lambda/2 \leq K/2 \) where \( K+1 \) is the number of the terms of the equation

\[
P_K(z) = 1 + z + \alpha_1z^n + \cdots + \alpha_{K-1}z^{n_{K-1}} = 0,
\]

and since the circle of centre \(-\lambda/2\) and radius \(\lambda/2\) is covered by the circle of center \(-K/2\) and radius \(K/2\), we have the theorem:

**Theorem.** The polynomial \( P_K(z) = 1 + z + \alpha_1z^n + \cdots + \alpha_{K-1}z^{n_{K-1}} \) of \( K+1 \) terms has at least one root within or on the circumference of a circle of centre \(-K/2\) and radius \(K/2\).

**Corollary.** The equation \( \alpha_0 + \alpha_1z + \alpha_2z^2 + \cdots + \alpha_mz^m = 0 \) has at least one root within or on the circumference of a circle of centre \( -(\alpha_0/\alpha_1) \cdot m/2 \) and radius \( |\alpha_0/\alpha_1| \cdot m/2 \).

If in the proof of the main theorem, the plane \( \Pi(x) \) is replaced by any closed half-plane containing the origin but not containing \(-\theta\), the reasoning is still valid and one obtains as a result the following:

**Theorem.** Let \( C \) be a closed circular disk containing on its boundary the points \( z = 0 \) and \( z = -\lambda \). Then \( C \) contains at least one root of \( P_K(z) \).

**References**


**Athens, Greece**