The purpose of this note is to present a purely geometrical result that appeared as a by-product of research in the theory of games. A preliminary draft had been written when it was discovered that a proof of the theorem had been published by I. Liberman [1] in 1943. However, the present proof seems to retain its interest, both for its conciseness and the fact that it exposes a connection between the geometry of convex sets and fixed-point theory that was motivated by game theory.

The result in question is:

**Theorem.** In order that a locally contractible compact set $X$ in $n$-dimensional Euclidean space be convex, it is necessary and sufficient that the set $X$ be contractible and that every support contact of $X$ be contractible.

By a *support contact* of $X$ is meant the intersection of $X$ with a supporting hyperplane, that is, the set of $x = (x_1, \ldots, x_n) \in X$ for which a given linear form $(a, x) = a_1x_1 + \cdots + a_nx_n$ assumes its maximum value $\alpha$; support contacts of convex sets are obviously convex. The necessity of the conditions is clear since a convex set $X$ can always be contracted to any point $x_0 \in X$ by the contraction $h(x, t) = tx_0 + (1-t)x$ where $x \in X$ and $0 \leq t \leq 1$. To prove the sufficiency of the conditions, assume that $X$ is not convex and let $C \supset X$ be the convex hull of $X$. Then there exists a point $p$ on the boundary of $C$ that does not lie in $X$ (otherwise $X$ would contain the boundary of $C$ which is a topological sphere, but not some point inside the sphere, contradicting the contractibility of $X$). Since $X$ is compact, $C$ is also and there exists a hyperplane supporting $C$ through $p$. Choose coordinates in the Euclidean space so that $p$ is the origin $0$ and the supporting hyperplane is given by $x_1 + \cdots + x_n = 0$, that is, $x_1 + \cdots + x_n \leq 0$ for all $x \in C$. Relative to this choice of coordinates, consider the zero-sum two-person game played thus:

**Move 1.** Player I chooses a point $x \in X$.

**Move 2.** Player II chooses an integer $i = 1, \ldots, n$.

**Payoff.** Player II pays Player I the amount $x_i$. 

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**Presented to the Society, October 24, 1953; received by the editors March 4, 1954.**

1 The preparation of this paper was sponsored (in part) by the Office of Scientific Research, USAF, and (in part) by the Logistics Project, ONR, Princeton University.

2 Numbers in brackets refer to the bibliography at the end of the paper.
If Player II plays the integer $i$ with probability $y_i$, where $y_i \geq 0$ and $y_1 + \cdots + y_n = 1$, then the expected payoff is

$$f(x, y) = x_1 y_1 + \cdots + x_n y_n.$$  

(1)

Since $x_1 + \cdots + x_n \leq 0$ for all $x \in X$ and $0 \notin X$, it is clear that every $x \in X$ has at least one negative coordinate and hence

$$\max_x \min_y f(x, y) = \max_x \min_i x_i < 0,$$

(2)

where the existence of the maximum is assured by the compactness of $X$. Thus the value of the game to Player I is negative. On the other hand, suppose Player II can choose probabilities $y_1, \ldots, y_n$ such that $\max_x f(x, y) = \alpha < 0$. Then $(\bar{y}, x) = \alpha$ is a supporting hyperplane for $X$ and hence for $C$. However $0 \in C$ and $(\bar{y}, 0) = 0 > \alpha$. Hence no such $\bar{y}$ exists and

$$\min_y \max_x f(x, y) \geq 0.$$  

(3)

Thus the value of the game to Player II is nonpositive, and the game does not have a value.

In a recent paper [2], Debreu has observed that fixed-point theorems of Eilenberg and Montgomery [3] or Begle [4] imply the following theorem on zero-sum two-person games:

Let $X$ and $Y$ be contractible and locally contractible compact sets in $n$-dimensional Euclidean space, and $f(x, y)$ a continuous real-valued function defined on $X \times Y$. If the sets

$$U_x = \{ y \in Y \mid f(\bar{x}, y) = \min_y f(\bar{x}, y) \},$$

$$V_y = \{ x \in X \mid f(x, \bar{y}) = \max_x f(x, \bar{y}) \}$$

are contractible for all $\bar{x} \in X$ and all $\bar{y} \in Y$, then

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

(4)

In the language of game theory, if $X$ and $Y$ are the sets of strategies and $f(x, y)$ the payoff function for a zero-sum two-person game, then this theorem asserts that the game has a value. For the game defined above, the sets $U_x$ are faces of the simplex $Y$ of probabilities $y_1, \ldots, y_n$ since the function $f$ is linear. On the other hand, the sets $V_y$ are support contacts of $X$ and hence are contractible by assumption. Therefore the game has a value, contradicting conclusions (2) and (3) above. The assumption that $X$ is not convex falls with them and
this completes the proof of the theorem.

Liberman's theorem is stronger than the result just proved in one important respect; it does not presuppose local contractibility. This condition entered the proof by way of the fixed-point theorems and it does not seem possible to eliminate it directly without a stronger theorem than seems to exist in the literature. Indeed, it would require the following conjectural statement:

Let $X$ be a contractible compact set in $n$-dimensional Euclidean space and $T$ an upper semi-continuous mapping which assigns to each point $x \in X$ an acyclic subset $T(x) \subseteq X$. Then $T$ has a fixed point, i.e., for some $x$, $T(x)$ contains $x$.

Although the special case of a single-valued mapping is a well-known and long-standing problem (an example of Borsuk [5] shows that it is not enough to assume that $X$ is an acyclic compact set), the validity of Liberman's theorem seems to indicate possible progress in this direction.

(Added March 30, 1954. The discovery by S. Kinoshita of a contractible continuum in 3-dimensional Euclidean space without fixed point property (Fund. Math. vol. 40 (1953) pp. 96–98) means that Liberman's theorem cannot be proved directly by means of a fixed point theorem.)

**Bibliography**