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UNIVERSITY OF WISCONSIN

A NOTE ON CORRELATION FUNCTIONS AND STABILITY IN DYNAMICAL SYSTEMS[1]

C. R. PUTNAM

1. In the system of differential equations

$$x' = f(x), \tag{1}$$

in which $x = (x_1, \cdots, x_n)$ and $f(x) = (f_1, \cdots, f_n)$ is of class C^1, suppose that

$$\operatorname{div} f = \partial f_1/\partial x_1 + \cdots + \partial f_n/\partial x_n = 0 \tag{2}$$

throughout a compact, invariant space, Ω, of (finite) positive v-measure. It will be supposed that Ω is the closure of an open set and that v refers to the ordinary n-dimensional Lebesgue volume measure in the x-space consisting of points $P = x$.

If $g(t)$ denotes any function of class (L) on every finite t-interval of $-\infty < t < \infty$, then $M_t[g(t)]$ will be defined as

$$M_t[g(t)] = \lim\, (2T)^{-1} \int_{-T}^{T} g(t)dt, \qquad T \to \infty,$$

in case the limit exists. In view of the measure-preserving assumption (2), it follows from the Birkhoff ergodic theorem that if $f = f(P)$ denotes any function of class (L^2) on Ω, then the correlation function

$$c_P(s) = M_t[f(P_{t+s})\bar{f}(P_t)] \tag{3}$$

(which depends on f) exists as a continuous function of s on $-\infty < s < \infty$ for almost all points P of Ω, where $P = x(0)$ and $P_t = x(t)$ denotes

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a solution of (1) through the point P; cf. [6, p. 805]. Furthermore, according to a result of Wiener, there exists a uniquely determined monotone function $\sigma_P(\lambda)=\sigma_P^f(\lambda)$ satisfying $\sigma_P(-\infty)=0$, $\sigma_P(\lambda-0)=\sigma_P(\lambda)$ such that

$$(4)\qquad c_P(s) = \int_{-\infty}^{\infty} e^{is\lambda}d\sigma_P(\lambda)$$

holds for all points P for which (3) exists (hence, almost everywhere on Ω); cf. [6, p. 797]. Next, let $F(s)$ be defined by

$$(5)\qquad F(s) = \int_\Omega f(P_s)\bar{f}(P)dv, \qquad -\infty < s < \infty.$$

It follows from an integration over Ω of both sides of the equation (3) that $F(s)=\int_\Omega c_P(s)dv$, and hence, from (4), that

$$(6)\qquad F(s) = \int_{-\infty}^{\infty} e^{is\lambda}d\Sigma(\lambda),$$

where $\Sigma(\lambda)$ denotes the monotone function

$$(7)\qquad \Sigma(\lambda) = \int_\Omega \sigma_P(\lambda)dv;$$

cf. [6, pp. 813–814]. It should be noted that the functions (4) and (6) are almost periodic (B^2); cf. [6, p. 798], and [1].

A solution $x=x(t)$ of (1) will be called stable (with respect to Ω) if stability is meant in the Minding-Dirichlet or Liapounoff sense; thus, for every $\epsilon>0$ there exists a $\delta=\delta_\epsilon>0$ such that $|x(t)-y(t)|<\epsilon$, $-\infty<t<\infty$, whenever $y(t)$ denotes a solution of (1) lying in Ω and satisfying $|x(0)-y(0)|<\delta$. See [2] and [4].

The following theorem will be proved:

(*) *Let $f(P)$ denote a continuous function on the invariant space, Ω, of the type considered above (so that Ω is compact and identical with the closure of an open set, and hence is of positive measure), and suppose that P_t^* denotes a stable path of* (1). *Then, if $f(P_t^*)$ is almost periodic* (B^2), *its frequencies are contained in the set of frequencies of the almost periodic* (B^2) *function $F(s)$ defined by* (5), *that is, in the point spectrum of the function $\Sigma(\lambda)$, which depends on f but is independent of the particular stable path P_t^* under consideration, defined by* (6) (*or* (7)).

As a consequence of (*) one obtains the following

COROLLARY. *If, in addition to the assumptions of* (*), *it is assumed that the function $F(S)$ of* (5) *satisfies*

$$(8) \qquad F(s) \to 0 \qquad \text{as } |s| \to \infty,$$

and if $f(P_t^*)$ *is uniformly almost periodic (in the sense of Bohr), then* $f(P_t^*) \equiv 0$ *for* $-\infty < t < \infty$.

In fact, according to [6, p. 797], $M_s[e^{-is\lambda}F(s)]$ exists and equals $\Sigma(\lambda+0)-\Sigma(\lambda-0)$. But, by (8), this difference is zero and the assertion of the last corollary now readily follows from (*).

It is interesting to note that, under the assumptions of (*) relating to $f(P)$ and Ω, the limit

$$(9) \qquad \lim T^{-1}\int_0^T f(P_t^*)dt, \qquad \text{as } T \to \infty \text{ (or } -\infty),$$

exists whenever P_t^* is a stable (not necessarily almost periodic (B^2)) path [4]. Moreover, relation (8) (which represents a type of statistical independence [3, p. 67]) implies that f is ergodic, so that $\lim T^{-1}\int_0^T f(P_t)dt$, as $T\to\infty$ (or $-\infty$), exists, and is zero (hence $\int_\Omega f(p)dv=0$), almost everywhere on Ω; loc. cit. pp. 68–69. Thus, since P_t^* is stable, each of the limits of (9) is zero (cf. the argument used in [4]) and so $M_t[f(P_t^*)]=0$. According to the theorem (*) however, if, in addition to (8), it is also assumed that $f(P_t^*)$ is uniformly almost periodic (in the sense of Bohr), then necessarily $f(P_t^*)\equiv 0$, $-\infty<t<\infty$.

It is noteworthy that in (*) it is *not* assumed that the stable path, say P_t^*, is dense in the space Ω. Thus, it is surprising that the frequencies of each of the n components $x_k=x_k(P_t^*)$ of a stable almost periodic (B^2) path are already determined by (more precisely, must be contained in) the set of frequencies of the corresponding almost periodic (B^2) function $F_k(s)=\int_\Omega x_k(P_s)x_k(P)dv$, each being an integral taken over the entire "phase space" Ω.

The theorem (*) is similar, in certain respects, to that given by Wiener and Wintner [6, p. 814]. In the present case, however, while the content of the assertion is restricted to continuous functions $f(P)$ on stable paths P_t^*, the hypothesis involves only the function $f(P)$ (and not the entire flow itself, P_t, of the system (1)).

For other results dealing with the connection between stability and almost periodicity, see [2] and [4].

2. **Proof of** (*). If λ is a fixed (real) number, then $M_t[e^{-i\lambda t}f(P_t)]$ exists almost everywhere on Ω; cf. [6, p. 804]. Since P_t^* is stable, it readily follows that $M_t[e^{-i\lambda t}f(P_t^*)]$ exists for every λ. In fact, if this limit failed to exist for the stable path P_t^*, it would also fail to exist for all points P sufficiently near P^* (hence, in view of the assumptions

on Ω, on a subset of Ω of positive v-measure) in contradiction with the Birkhoff ergodic theorem; cf. [4].

Next, let $\lambda=\lambda_0$ belong to the frequency spectrum of $f(P_t^*)$, so that $M_t[e^{-i\lambda_0 t}f(P_t^*)]\neq 0$. Then $M_t[e^{-i\lambda_0 t}f(P_t)]\neq 0$ (again, cf. [4]) for almost all points P in some neighborhood of P^*. In view of the inequality $|M_t[e^{-i\lambda_0 t}f(P_t)]|^2\leqq\sigma_P(\lambda_0+0)-\sigma_P(\lambda_0-0)$ ([6, p. 799]; note that $f(P_t)$ need not be almost periodic (B^2)), it follows that $\sigma_P(\lambda_0+0)-\sigma_P(\lambda_0-0)>0$ on some of set positive v-measure. According to (7), however, $\Sigma(\lambda+0)-\Sigma(\lambda-0)=\int_\Omega[\sigma_P(\lambda+0)-\sigma_P(\lambda-0)]dv$, and so $\Sigma(\lambda_0+0)-\Sigma(\lambda_0-0)>0$; thus, $\lambda=\lambda_0$ is in the point spectrum of $\Sigma(\lambda)$. This completes the proof of (*).

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Purdue University