there exists an element \( y \) of \( G \) such that \( Fy = x' \). Thus the distinct elements \( x, y \) of \( H \) are mapped by \( F \) into the same element \( x' \) of \( V_T \), contrary to condition (ii).

**References**


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**A NOTE ON A RECENT RESULT IN SUMMABILITY THEORY**

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In a recent paper\(^1\) by A. Mary Tropper, the following theorem is given:

**Theorem.** In order that the regular\(^2\) normal\(^3\) matrix \( A \) shall sum a bounded divergent sequence, it is sufficient that

(a) its unique reciprocal \( B \) shall not be regular and

(b) there exists a normal matrix \( Q \) with\(^4\) \( \| Q \| < \infty \) whose columns are all null sequences, such that the matrix \( C = BQ \) has bounded columns and \( \| C \| = \infty \).

The author points out that (a) is also a necessary condition, but does not prove the necessity of condition (b); the object of this note is to prove that condition (b) is necessary.

The proof given of the theorem quoted above holds if the \( K_r \),

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\(^2\) A regular matrix is one which satisfies \( \lim_{n \to \infty} \sum_{k=1}^{n} a_{n,k}x_k = \lim_{k \to \infty} x_k \) whenever the latter limit exists.
\(^3\) A normal matrix is a lower-semi matrix which has no zero elements in the leading diagonal. Such a matrix has a unique two-sided reciprocal; see e.g., R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, 1950, p. 19.
\(^4\) For an infinite matrix \( M, \| M \| = \sup_n \sum_k |m_{n,k}| \) is called the \( K_r \) bound of the matrix; a matrix whose \( K_r \) bound is finite is called a \( K_r \) matrix (ibid. pp. 25, 29).
matrix $Q$ of condition (b) is required to be a lower-semi, but not necessarily a normal, matrix; the necessity of condition (b) will first be proved under these conditions, and then it will be shown that the existence of a normal $K_r$ matrix $Q$ is also necessary.

Suppose, therefore, that $A$ sums the bounded divergent sequence $\{x_i\}$ to $x$. Without loss of generality, we may suppose $x=0$, since, if $x \neq 0$, $A$ sums the bounded divergent sequence $\{x_i-x\}$ to 0.

Let $C^{(1)}$ be the normal matrix defined by

$$c_{i,j}^{(1)} = x_i (j < i); \quad c_{i,i}^{(1)} = 1; \quad c_{i,j}^{(1)} = 0 (j > i).$$

Any given column of $C^{(1)}$ is thus the sum of $\{x_i\}$ and a null sequence; hence its $A$-transform tends to 0, i.e., the columns of $Q^{(1)} = AC^{(1)}$ all form null sequences.

Also,

$$(1) \quad \left| q_{r,s}^{(1)} \right| = \left| \sum_{i=1}^{r} a_{r,i} c_{i,s}^{(1)} \right| \leq \sup \limits_{i} \left| c_{i,s}^{(1)} \right| \sup \limits_{r} \sum_{i=1}^{r} \left| a_{r,i} \right| < K \quad \text{for all } r, s.$$

Now a diagonal matrix $D$, whose diagonal elements are all either 0 or 1, can be chosen in such a way that $D$ contains an infinity of 1's and $Q = Q^{(1)} D$ is a $K_r$ matrix. For, to choose such a matrix, let the values of $n$ for which $d_n = 1$ be denoted by $n_1, n_2, \ldots$, and suppose that $n_1, n_2, \ldots, n_p$ have been chosen such that

$$(2) \quad \sum_{s=1}^{n_r} \left| q_{r,s} \right| = \sum_{k=1}^{p} \left| q_{r,n_k}^{(1)} \right| \leq 2K \quad \text{whenever } r \geq n_v (v = 1, 2, \ldots, p).$$

Since the columns of $Q^{(1)}$ tend to 0, we may choose $n_{p+1} > n_p$ such that

$$\sum_{k=1}^{p} \left| q_{r,n_k}^{(1)} \right| \leq K \quad (r \geq n_{p+1}),$$

and hence, by (1),

$$\sum_{s=1}^{n_{p+1}} \left| q_{r,s} \right| \leq 2K \quad (r \geq n_{p+1}).$$

We may begin this construction by taking $n_1 = 1$, and it continues indefinitely, so that $D$ contains an infinity of 1's.

We now show that with $D$ so defined, $\|Q\| < \infty$. Given any value of $r$, there exists $p$ such that $n_p \leq r < n_{p+1}$; then

$$\sum_{s=1}^{r} \left| q_{r,s} \right| = \sum_{k=1}^{p} \left| q_{r,n_k}^{(1)} \right| \leq 2K,$$
by (2), which is what is required.

Also, the columns of \( Q \) form null sequences, since they are either columns of \( Q^{(1)} \), or they consist entirely of zeros.

Again \( C = BQ = BQ^{(1)} D = BA C^{(1)} D = C^{(1)} D \), where \( B \) is the unique two-sided reciprocal of \( A \). (The products are associative, since the matrices are all lower-semi.) Hence,

\[
\sum_{j=1}^{i} |c_{i,j}| = \sum_{k=1}^{\lambda_i} |c_{i,n_k}^{(1)}|,
\]

where \( \lambda_i \) is the number of 1's in the first \( i \) columns of \( D \); thus

\[
\sum_{j=1}^{i} |c_{i,j}| \geq (\lambda_i - 1) |x_i|.
\]

But \( \lim_{i \to \infty} \lambda_i = \infty \), and \( \lim \sup_{i \to \infty} |x_i| > 0 \), since \( \{x_i\} \) is divergent. Hence \( \lim \sup_{i \to \infty} \sum_{j=1}^{i} |c_{i,j}| = \infty \), so that \( \|C\| = \infty \).

Moreover, the columns of \( C \) are bounded, for they are either columns of \( C^{(1)} \), or they consist entirely of zeros.

This completes the proof in the case where \( Q \) is required to be a lower-semi, but not necessarily normal, matrix.

To show that a normal matrix \( Q \) also exists, we note that \( C^{(1)} \) and \( Q^{(1)} \) are normal matrices, and that \( I - D \) is a diagonal matrix of 0's and 1's whose nonzero elements occur only in those columns of \( D \) which consist entirely of zeros. Hence \( \overline{C} = C + I - D = C^{(1)} D + I - D \) is a normal matrix with bounded columns, and \( \|\overline{C}\| = \infty \).

The corresponding \( Q \), i.e., \( \overline{Q} = A \overline{C} \), is normal (being the product of normal matrices), and, since \( \overline{Q} = AC^{(1)} D + A (I - D) = Q^{(1)} D + A (I - D) \), we have \( \|\overline{Q}\| \leq \|Q\| + \|A\| < \infty \); also the columns of \( \overline{Q} \), which are all either columns of \( Q^{(1)} \) or columns of \( A \), form null sequences.

Hence the existence of a normal matrix \( Q \) with the required properties is proved.

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