$F(G) = V_T$, there exists an element $y$ of $G$ such that $Fy = x'$. Thus the distinct elements $x, y$ of $H$ are mapped by $F$ into the same element $x'$ of $V_T$, contrary to condition (ii).

References


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A NOTE ON A RECENT RESULT IN SUMMABILITY THEORY

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In a recent paper¹ by A. Mary Tropper, the following theorem is given:

Theorem. In order that the regular² normal³ matrix $A$ shall sum a bounded divergent sequence, it is sufficient that

(a) its unique reciprocal $B$ shall not be regular and
(b) there exists a normal matrix $Q$ with⁴ $\|Q\| < \infty$ whose columns are all null sequences, such that the matrix $C = BQ$ has bounded columns and $\|C\| = \infty$.

The author points out that (a) is also a necessary condition, but does not prove the necessity of condition (b); the object of this note is to prove that condition (b) is necessary.

The proof given of the theorem quoted above holds if the $K_r$.

² A regular matrix is one which satisfies $\lim_{n \to \infty} \sum_{k=1}^{n} a_{n,k}x_k = \lim_{k \to \infty} x_k$, whenever the latter limit exists.
³ A normal matrix is a lower-semi matrix which has no zero elements in the leading diagonal. Such a matrix has a unique two-sided reciprocal; see e.g., R. G. Cooke, Infinite matrices and sequence spaces, Macmillan, 1950, p. 19.
⁴ For an infinite matrix $M$, $\|M\| = \sup_n \sum_k |m_{n,k}|$ is called the $K_r$ bound of the matrix; a matrix whose $K_r$ bound is finite is called a $K_r$ matrix (ibid. pp. 25, 29).
matrix $Q$ of condition (b) is required to be a lower-semi, but not necessarily a normal, matrix; the necessity of condition (b) will first be proved under these conditions, and then it will be shown that the existence of a normal $K_T$ matrix $Q$ is also necessary.

Suppose, therefore, that $A$ sums the bounded divergent sequence $\{x_i\}$ to $x$. Without loss of generality, we may suppose $x = 0$, since, if $x \neq 0$, $A$ sums the bounded divergent sequence $\{x_i - x\}$ to 0.

Let $C^{(1)}$ be the normal matrix defined by

$$c_{i,j}^{(1)} = x_i \ (j < i); \quad c_{1,j}^{(1)} = 1; \quad c_{i,i}^{(1)} = 0 \ (j > i).$$

Any given column of $C^{(1)}$ is thus the sum of $\{x_i\}$ and a null sequence; hence its $A$-transform tends to 0, i.e., the columns of $Q^{(1)} = AC^{(1)}$ all form null sequences.

Also,

$$(1) \quad |q_{r,s}^{(1)}| = \left| \sum_{i=1}^{r} a_{r,i} c_{i,s}^{(1)} \right| \leq \sup_{i} |c_{i,s}^{(1)}| \sup_{r} \left| \sum_{i=1}^{r} a_{r,i} \right| < K \quad \text{for all } r, s.$$

Now a diagonal matrix $D$, whose diagonal elements are all either 0 or 1, can be chosen in such a way that $D$ contains an infinity of 1's and $Q = Q^{(1)}D$ is a $K_T$ matrix. For, to choose such a matrix, let the values of $n$ for which $d_n = 1$ be denoted by $n_1, n_2, \ldots$, and suppose that $n_1, n_2, \ldots, n_p$ have been chosen such that

$$(2) \quad \sum_{s=1}^{n_{\nu}} |q_{r,s}^{(1)}| = \sum_{k=1}^{r} |q_{r,n_{\nu}}^{(1)}| \leq 2K \quad \text{whenever } r \geq n_{\nu} \ (\nu = 1, 2, \ldots, \rho).$$

Since the columns of $Q^{(1)}$ tend to 0, we may choose $n_{\rho+1} > n_{\rho}$ such that

$$\sum_{k=1}^{\rho} |q_{r,n_k}^{(1)}| \leq K \quad (r \geq n_{\rho+1}),$$

and hence, by (1),

$$\sum_{s=1}^{n_{\rho+1}} |q_{r,s}| \leq 2K \quad (r \geq n_{\rho+1}).$$

We may begin this construction by taking $n_1 = 1$, and it continues indefinitely, so that $D$ contains an infinity of 1's.

We now show that with $D$ so defined, $\|Q\| < \infty$. Given any value of $r$, there exists $p$ such that $n_p \leq r < n_{p+1}$; then

$$\sum_{s=1}^{r} |q_{r,s}| = \sum_{k=1}^{p} |q_{r,n_k}^{(1)}| \leq 2K,$$
by (2), which is what is required.

Also, the columns of $Q$ form null sequences, since they are either columns of $Q^{(1)}$, or they consist entirely of zeros.

Again $C = BQ = BQ^{(1)}D = BAC^{(1)}D = C^{(1)}D$, where $B$ is the unique two-sided reciprocal of $A$. (The products are associative, since the matrices are all lower-semi.) Hence,

$$
\sum_{j=1}^{i} |c_{i,j}| = \sum_{k=1}^{\lambda_i} |c_{i,n_k}^{(1)}|,
$$

where $\lambda_i$ is the number of 1’s in the first $i$ columns of $D$; thus

$$
\sum_{j=1}^{i} |c_{i,j}| \geq (\lambda_i - 1) |x_i|.
$$

But $\lim_{i \to \infty} \lambda_i = \infty$, and $\lim \sup_{i \to \infty} |x_i| > 0$, since $\{x_i\}$ is divergent. Hence $\lim \sup_{i \to \infty} \sum_{j=1}^{i} |c_{i,j}| = \infty$, so that $\|C\| = \infty$.

Moreover, the columns of $C$ are bounded, for they are either columns of $C^{(1)}$, or they consist entirely of zeros.

This completes the proof in the case where $Q$ is required to be a lower-semi, but not necessarily normal, matrix.

To show that a normal matrix $Q$ also exists, we note that $C^{(1)}$ and $Q^{(1)}$ are normal matrices, and that $I - D$ is a diagonal matrix of 0’s and 1’s whose nonzero elements occur only in those columns of $D$ which consist entirely of zeros. Hence $\bar{C} \equiv C + I - D = C^{(1)}D + I - D$ is a normal matrix with bounded columns, and $\|\bar{C}\| = \infty$.

The corresponding $Q$, i.e., $\bar{Q} = A\bar{C}$, is normal (being the product of normal matrices), and, since $\bar{Q} = AC^{(1)}D + A(I - D) = Q^{(1)}D + A(I - D)$, we have $\|\bar{Q}\| \leq \|Q\| + \|A\| < \infty$; also the columns of $\bar{Q}$, which are all either columns of $Q^{(1)}$ or columns of $A$, form null sequences.

Hence the existence of a normal matrix $Q$ with the required properties is proved.

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