A NON-EXCEPTIONAL ELEMENT OF WIENER SPACE

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Let $C$ denote the space of functions $x(t)$, continuous on $0 \leq t \leq 1$ with $x(0) = 0$. For each element of $C$ let

$$
\sigma_n(x) = \sum_{j=1}^{2^n} \left[ x\left( \frac{j}{2^n} \right) - x\left( \frac{j-1}{2^n} \right) \right]^2.
$$

Cameron and Martin [1] have shown that for almost all elements of $C$, in the sense of Wiener measure, the limit

$$
\int_0^1 |dx(t)|^2 = \lim_{n \to \infty} \sigma_n(x)
$$

exists and has the value $1/2$.

The purpose of this note is to construct an explicit example of an element of $C$ for which the above limit is equal to $1/2$. The example is of interest, since for most "ordinary" functions the value of the limit is zero. If $x(t)$ is of bounded variation on $0 \leq t \leq 1$, the limit is zero. If $x(t)$ satisfies for $0 \leq t \leq 1$ a uniform Lipschitz condition of order $\alpha$, where $\alpha$ is greater than $1/2$, the limit is zero, whereas it is possible for such a function to be everywhere nondifferentiable [2].

The present construction is a modification of one given by the author [2]. In the notation of [2] we use $A = \alpha = 1/2$. Specifically let $g(t, h)$ be periodic of period $2h$, equal to zero for even multiples of $h$, equal to one for odd multiples of $h$, and linear between. Let $x_0(t)$ be defined by

$$
x_0(t) = \frac{1}{2} \sum_{m=1}^{\infty} 2^{-m/2} g(t, 2^{-m}).
$$

$x_0(0) = 0$, and $x_0(t)$ is continuous on $0 \leq t \leq 1$ by uniform convergence of the series. Thus $x_0(t) \in C$. We now show that

$$
\int_0^1 |dx_0(t)|^2 = \frac{1}{2}.
$$

If the interval $0 \leq t \leq 1$ is divided into $2^n$ equal parts, the increments of $x_0(t)$ are the $2^n$ different values that can be obtained from all possible combinations of signs in the expression

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1 The author has been informed that this result was found originally by P. Lévy.
\[
\frac{1}{2^n} \left( \pm 2^{n/2} \pm 2^{(n-1)/2} \pm \cdots \pm 2^{1/2} \right).
\]

This follows from the fact that only the first \( n \) terms of the series for \( x_0(t) \) will have increments different from zero, while the directions of the increments of the first \( n \) terms will occur in all possible combinations. Now when we compute the sum of the squares of these \( 2^n \) increments, the cross-product terms occur as often positive as negative, so that only the squares of the terms contribute, and the sum of the squares of the increments is

\[
\sigma_n(x_0) = \frac{1}{4} 2^{-2n} 2^n (2^n + 2^{n-1} + \cdots + 2)
\]

\[
= \frac{1}{2} - \frac{1}{2^{n+1}}.
\]

Therefore

\[
\lim_{n \to \infty} \sigma_n(x_0) = \int_0^1 |dx_0(t)|^2 = \frac{1}{2}.
\]

**Bibliography**


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