CORRECTION TO "UNIQUENESS OF THE PROJECTIVE PLANE WITH 57 POINTS"

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Professor Gunter Pickert has pointed out to me that there is an error in my paper [Proc. Amer. Math. Soc. vol. 4 (1953) pp. 912–916]. The equation on page 915 which reads $12 + 3s + 2t = 24$ should read $12 + 3s + 2t + u = 24$. This invalidates my conclusion $u = 0$ from which I deduce $U = 0$, which is necessary for the rest of the paper. I give here a new proof that $U = 0$.

We must show that in a 57 point plane a line containing points $ACC$ is not possible, or that $U = 0$. The proof is by showing that if there is a line $ACC$ we reach a contradiction. We may take $ACC$ as $A_1C_4C_6$ by numbering points appropriately. We now reletter by the following rule:

\[
\begin{array}{cccccccc}
A_1 & A_2 & A_3 & A_4 & B_1 & B_2 & B_3 & C_4 & C_6 \\
A_1 & B_1 & B_5 & A_0 & B_4 & B_2 & A_2 & B_3 & B_6
\end{array}
\]

and obtain the following configuration:

\[
\begin{array}{cccc}
A_0 & B_1 & B_2 & B_3 \\
A_0 & B_4 & B_5 & B_6 \\
A_0 & A_1 & A_2 \\
A_1 & B_1 & B_4 \\
A_1 & B_2 & B_6 \\
A_1 & B_3 & B_6 \\
A_2 & B_1 & B_5 \\
A_2 & B_2 & B_6 \\
A_2 & B_3 & B_4
\end{array}
\]
The remaining points in these lines must be filled in according to the following pattern:

\[
\begin{array}{cccccccccc}
A_0 & B_1 & B_2 & B_3 & C_1 & C_2 & C_3 & C_4 \\
A_0 & B_4 & B_5 & B_6 & C_5 & C_6 & C_7 & C_8 & A & 3 \\
L- & A_0 & A_1 & A_2 & D_1 & D_2 & D_3 & D_4 & D_5 & B & 6 \\
A_1 & B_1 & B_4 & E_1 & F_1 & F_2 & F_3 & F_4 & C & 8 \\
A_1 & B_2 & B_6 & E_2 & F_5 & F_6 & F_7 & F_8 & D & 5 \\
A_1 & B_3 & B_6 & E_3 & F_9 & F_{10} & F_{11} & F_{12} & E & 3 \\
A_2 & B_1 & B_5 & E_3 & F_{13} & F_{14} & F_{15} & F_{16} & F & 24 \\
L- & A_2 & B_2 & B_6 & E_1 & F_{17} & F_{18} & F_{19} & F_{20} & G & 8 \\
A_2 & B_3 & B_4 & E_2 & F_{21} & F_{22} & F_{23} & F_{24} & & 57 \\
\end{array}
\]

The points \( E_1, E_2, E_3 \) provide the necessary remaining intersection points for these lines. All points as given here are necessarily distinct and there will be 8 remaining points which will be designated by the letter \( G \).

Remaining lines are of the following patterns:

\[
\begin{array}{cccccccccccc}
P- & A_0 & E & E & E & G & G & G \\
Q- & A_0 & E & E & F & F & G & G & G & \text{Remaining } A_0 \\
R- & A_0 & E & F & F & F & F & G & G & P + Q + R + S = 5. \\
S- & A_0 & F & F & F & F & F & G & G \\
N- & A & C & C & F & F & F & G & G & \text{Remaining } A \text{'s} \\
U- & B & B & D & E & G & G & G & \text{Remaining } BB \text{'s} \\
V- & B & B & D & F & F & G & G & G & U + V = 3. \\
W- & B & C & D & E & F & F & G & G & \text{Remaining } B \text{'s} \\
X- & B & C & D & F & F & F & F & G & W + X = 24. \\
Y- & C & C & D & E & E & E & E & G & \text{Remaining } CC \text{'s} \\
Z- & C & C & D & E & E & F & F & G & Y + Z + T = 8. \\
T- & C & C & D & E & E & F & F & F & \text{ } \\
\end{array}
\]

Counting \( BE \)'s we find \( 12 + 2U + W = 18 \) whence \( W = 6 - 2U = 2V \), and \( X = 24 - W = 24 - 2V \). Counting \( CE \)'s we find \( W + 6Y + 4Z + 2T = 24 \) and combining with \( Y + Z + T = 8 \) and \( W = 2V \) we obtain \( Z = 4 - V - 2Y \), \( T = 4 + V + Y \). Now count \( EE \)'s \( 3P + Q + 3Y + Z = 3 \).
Hence if $P = 1$ then $Y = Z = 0$ whence $V = 4$ in conflict with $U + V = 3$. Hence we must have $P = 0$.

Let us count occurrences of a point $C$ on the lines:

Total occurrences: $3 + w + x + y + z + t = 8$ (1 on $L$, 2 on $N$'s)

$CB$: $3 + w + x = 6$,

$CE$: $w + 3y + 2z + t = 3$.

Combining these we find $w + 2y + z = 1$. Hence $y = 0$ and since there is no $C$ on any $Y$ we must have $Y = 0$.

Let us now count occurrences of an $F$ on lines. It is on 1 $L$ and 1 $N$.

Total: $2 + q + r + s + v + w + x + z + t = 8$,

$FE$: $1 + 2q + r + w + 2z + t = 3$,

$FC$: $2 + w + x + 2z + 2t = 8$.

Now if $q = 1$, from the second equation we have $r = w = z = t = 0$, whence from the third equation $x = 6$. But $q = 1$, $x = 6$ conflicts with the first equation. Hence $q = 0$ and so $Q = 0$.

We use two more counts on the number of lines:

$A_6E$: $3P + 2Q + R = 3$

$GG$: $6P + 3Q + R + 8 + 6U + 3V + W + Y = 28$.

Since $P = Q = 0$, the first of these yields $R = 3$. Putting this in the second and also $W = 2V$, $Y = 0$, we find

$$6U + 5V = 17.$$ Since $U + V = 3$, we have $U = 2$, $V = 1$. Now with the relations above, we can determine the number of lines in each category:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$S$</th>
<th>$N$</th>
<th>$U$</th>
<th>$V$</th>
<th>$W$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$Z$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>22</td>
<td>0</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

We reach our final contradiction by showing that there is no point which can be the intersection point of the two lines of category $U$.

$$U - \quad B \quad B \quad D \quad E \quad G \quad G \quad G \quad G.$$ The remaining $BB$ pairs in $U$ and $V$ are $B_1B_6$, $B_2B_4$, and $B_3B_5$ and so the $U$ intersection is not a $B$. Now consider occurrences of a $D$ on lines. It is on 1 $L$.

Total: $1 + u + v + w + x + z + t = 8$.

$DF$: $2v + 2w + 4x + 2z + 4t = 24$. 

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Combining, $x + t = 5 + u$. But if $u = 2$, then $x + t = 7$ and these values conflict with the first equation. Hence we cannot have $u = 2$ and so the two $U$ lines do not intersect in a $D$. Now count occurrences of an $E$.

$EA_0$: \[ r = 1, \]

$EG$: \[ 2r + 4u + 2w + z = 8. \]

Here we cannot have $u = 2$ and the two $U$'s do not intersect in an $E$.

For occurrences of a $G$ we note that $u = 2$.

$GG$: \[ r + n + 3u + 2v + w = 7. \]

This is not possible with $n = 2$, $u = 2$. Hence the two $U$ lines do not intersect in a $G$. Since there is no point which can be the intersection of the two $U$'s, we have reached a final contradiction.

Professor Pickert further inquired the reason for the assertion on page 916 that a quadrilateral which did not generate a 7 point subplane must generate the whole 57 point plane. This depends on a result which appears to be well known, though not explicitly given in the literature. I give here a refinement due to Professor R. Bruck.

**Lemma.** If a plane with $n + 1$ points on a line has a proper subplane with $m + 1$ points on a line, then $n \geq m^2$. If $n \neq m^2$, then $n \geq m^2 + m$.

**Proof.** Let $L$ be a line of the subplane $\pi$ and $P$ a point on $L$ but not in $\pi$. Joining $P$ to the $m^2$ point of $\pi$ not on $L$ we have $m^2$ further lines through $P$. These must be distinct since otherwise $P$ would be the intersection of two lines of $\pi$, and hence a point of $\pi$. This gives $m^2 + 1$ lines through $P$ and so $n \geq m^2$. If there are any more lines through $P$, then there is a line $K$ through $P$ containing no point of $\pi$. Hence the intersections of $K$ with the $m^2 + m + 1$ lines of $\pi$ are all distinct and so $n \geq m^2 + m$.

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