ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

S. B. MYERS

1. Let $M$ be a compact differentiable manifold of class $C^r$, $1 \leq r < \infty$, and let $C^r(M)$ be the space of all real functions of class $C^r$ on $M$. In addition to the obvious algebraic structure of $C^r(M)$, we shall use a normed algebra structure, the norm being obtained by introducing a Riemannian metric on $M$. The main results (Theorems 1 and 3) will be stated and proved in this section, using lemmas on differentiable manifolds which will be proved in the following section. Theorem 1 is straightforward, Theorem 3 is more difficult.

**Theorem 1.** If $M$ is a compact differentiable manifold of class $C^r$, then $C^r(M)$ as an algebra determines the $C^r$ structure of $M$; i.e., if $C^r(M)$ is isomorphic to $C^r(N)$, where $M$ and $N$ are compact differentiable manifolds of class $C^r$, then there is a differentiable homeomorphism of class $C^r$ of $M$ onto $N$ with an inverse of class $C^r$.

**Proof.** $C^r(M)$ and $C^r(N)$ are inverse-closed (as subspaces of the spaces of all continuous functions on $M$ and $N$ respectively). Also, by Lemma 1, they are separating. It follows that the space $X$ of maximal ideals of $C^r(M)$ under the standard weak topology is homeomorphic to $M$, and the space $Y$ of maximal ideals of $C^r(N)$ is homeomorphic to $N$. But there is a homeomorphism of $X$ onto $Y$ because of the assumed isomorphism $I(C^r(M)) = C^r(N)$, and hence a homeomorphism $H(M) = N$; furthermore, if $f \in C^r(M)$ and $F \in C^r(N)$, then $(I(f))(y) = f(H^{-1}(y))$ and $(I^{-1}(F))(x) = F(H(x))$. Therefore, by Lemma 2, $H$ and $H^{-1}$ are differentiable homeomorphisms of class $C^r$.

**Theorem 2.** Let $M$ be a compact differentiable manifold of class $C^r$, provided with a Riemannian metric tensor $g_{ij}$ of class $C^{r-1}$. For $f \in C^r(M)$ define

$$||f|| = \max_{x \in M} |f(x)| + \max_{x \in M} \left( g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2}.$$ 

Then $C^r(M)$ becomes a real, commutative, semi-simple, normed algebra with unit; if $r = 1$, $C^r(M)$ is a Banach algebra.

Presented to the Society, May 1, 1954; received by the editors March 31, 1954.

1 This work was partially supported by the Office of Naval Research.

2 From the Gelfand theory. See, for example, Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, 1953, p. 55.

3 The summation convention of tensor analysis is used throughout.
PROOF. As is well known, at each point \( x \in M \),
\[
\left( g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2} = \max \left| \frac{\partial f}{\partial x^i} \eta^i \right|,
\]
the max being taken over the unit contravariant vectors \( \eta \) at \( x \). It easily follows that \( \| f_1 + f_2 \| \leq \| f_1 \| + \| f_2 \| \). Also,
\[
\| f_1 f_2 \| = \max \left| f_1(x) \right| \left| f_2(x) \right| + \max_{x \in M, \eta \text{ at } x} \left| \left( f_1 \frac{\partial f_2}{\partial x^i} + f_2 \frac{\partial f_1}{\partial x^i} \right) \eta^i \right|
\]
\[
\leq \max \left| f_1(x) \right| \max \left| f_2(x) \right|
\]
\[
+ \max \left| f_1(x) \right| \max \left| \frac{\partial f_2}{\partial x^i} \eta^i \right| + \max \left| f_2(x) \right| \max \left| \frac{\partial f_1}{\partial x^i} \eta^i \right|
\]
\[
\leq \| f_1 \| \| f_2 \| .
\]

As for the completeness of \( C^r(M) \), let \( f^\alpha \) be a Cauchy sequence in \( C^r(M) \). Then \( f^\alpha(x) \) converges uniformly to a continuous function \( f(x) \), while \( (\partial f^\alpha/\partial x^i)\eta^i \) as a sequence of continuous functions on the tangent bundle \( M \) of unit contravariant vectors of \( M \) converges uniformly over \( M \) to a function \( F(\eta) \) continuous on \( M \). If \( x^i = x^i(s) \) is a \( C^1 \) curve on \( M \), with unit tangent vector \( \eta^i(s) = dx^i/ds \), then
\[
\int_0^s \frac{\partial f^\alpha}{\partial x^i} \eta^i(s) ds = f^\alpha(x(s)) - f^\alpha(x(0))
\]
converges to \( \int_0^s F(\eta(s)) ds \). Hence
\[
f(x(s)) - f(x(0)) = \int_0^s F(\eta(s)) ds
\]
so that \( df(x(s))/ds = F(\eta(s)) \) for every curve in \( M \). Hence \( f(x) \) is of class \( C^1 \), \( (\partial f/\partial x^i)\eta^i = F(\eta) \) for all \( \eta \in M \), and \( (\partial f^\alpha/\partial x^i)\eta^i \) converges uniformly over \( M \) to \( (\partial f/\partial x^i)\eta^i \). Thus \( C^1(M) \) is complete.

**Theorem 3.** Let \( M \) be a compact differentiable manifold of class \( C^r \), provided with a Riemannian tensor \( g_{ij} \) of class \( r - 1 \). Then \( C^r(M) \) as a normed algebra (under the norm of Theorem 2) determines the Riemannian structure of \( M \). More precisely, if as normed algebras \( C^r(M) \) and \( C^r(N) \) (\( M \) and \( N \) compact) are equivalent, there is an isometry of class \( C^r \) of \( M \) onto \( N \).

**Proof.** According to Theorem 1, there is a nonsingular \( C^r \)-homeomorphism \( H(M) = N \), which induces the equivalence \( I(C^r(M)) = C^r(N) \). Let \( y^i = y^i(x) \) be local equations of \( H \), and let \( I(f) = F \); then
If $\eta$ is a contravariant vector at $(x)$ and $\xi$ is the vector $(\partial y/\partial x^i)\eta^i$ at $y(x)$, then $(\partial f/\partial x^i)(x) = (\partial F/\partial y^i)\xi^i$. Regarding $\eta$ and $\xi$ as continuous linear functionals over $C^r(M)$ and $C^r(N)$ respectively according to the formulas $\eta(f) = (\partial f/\partial x^i)\eta^i$, $\xi(F) = (\partial F/\partial y^i)\xi^i$, then $\eta(f) = \xi(F)$. Thus if $I^*$ is the equivalence of the conjugate spaces of $C^r(M)$ and $C^r(N)$ induced by $I$, then $I^*(\eta) = \xi$. Hence $\|\eta\| = \|\xi\|$. We now show $\|\eta\| = (g_{ij}\eta^i\eta^j)^{1/2}$, i.e. $\|\eta\|$ equals the magnitude of $\eta$. Without loss of generality we assume $\eta$ is a unit vector. Now

$$
\|\eta\| = \sup_{f} \left\{ \frac{\eta(f)}{\|f\|} \right\} = \sup_{f} \frac{\|\eta(f)\|}{\max_{x \in M} \|f\| + \max_{x \in M} |\eta(f)|} \leq 1.
$$

According to Lemma 3, given any $\epsilon, \delta > 0$ there exists $f \in C^r(M)$ such that

$$
\eta(f) = 1, \quad \max_{x \in M} \left( g^{ii} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^i} \right)^{1/2} < 1 + \epsilon, \quad \max_{x \in M} |f(x)| < \delta.
$$

Therefore $1 \leq \|f\| < 1 + \epsilon + \delta$, so $\|\eta\| \geq 1$.

Thus we have shown $\|\eta\| = 1 = (g_{ij}\eta^i\eta^j)^{1/2}$. Similarly $\|\xi\| = (h_{ij}\xi^i\xi^j)^{1/2}$, where $h_{ij}$ is the metric tensor on $N$. It follows that $g_{ij}\eta^i\eta^j = h_{ij}\xi^i\xi^j$, so that $g_{ij}\eta^i\eta^j = h_{kl}(\partial y^k/\partial x^i)(\partial y^l/\partial x^j)\eta^i\eta^j$ for arbitrary tangent vector $\eta$ to $M$. Hence $g_{ij} = h_{kl}(\partial y^k/\partial x^i)(\partial y^l/\partial x^j)$, so that $H$ is an isometry of class $C^r$.

2. Lemmas on differentiable manifolds.

**Lemma 1.** If $M$ is a differentiable manifold of class $C^r$, and if $P_0 \in M$ and $K$ is a closed set not containing $P_0$, then there is an $f \in C^r(M)$ such that $f(P_0) = 1$, $f(P) = 0$ for $P \in K$.

**Proof.** If $(x)$ is an admissible coordinate system about $P_0$ with $P_0$ as origin, with $\sum x^i x^i < 4\delta^2$, and with range in the complement of $K$, then the following function $f$ is the required function:

$$
f = \left(1 - \frac{\sum x^i x^i}{\delta^2}\right)^{r+1} \quad \text{for} \quad \sum x^i x^i \leq 4\delta^2,
$$

$$
f = 0 \quad \text{for all other points of} \quad M.
$$

**Lemma 2.** If $H$ is a homeomorphism of a differentiable manifold $M$ of class $C^r$ onto a differentiable manifold $N$ of class $C^r$, and if for every $F \in C^r(N)$ the induced function $f$ defined by $f(x) = F(H(x))$ belongs to $C^r(M)$, then $H$ is a differentiable homeomorphism of class $C^r$.

**Proof.** Let $P \in M$, and let $(x)$ and $(y)$ be admissible coordinate
systems about \( P \) and \( H(P) \) respectively, the domain of \( (y) \) being \( \sum y^i y^i < 3 \). Let \( y^i = y^i(x) \) be local equations of \( H \). Define

\[
A(\theta) = \frac{\int_1^\theta [(t-1)(t-2)]^r dt}{\int_1^2 [(t-1)(t-2)]^r dt}.
\]

Then for each \( i \) the function \( F^i \) defined as follows is of class \( C^r \) over \( N \):

\[
F^i = y^i \quad \text{for} \quad \sum y^i y^i \leq 1,
\]

\[
F^i = y^i - y^i A(\sum y^i y^i) \quad \text{for} \quad 1 \leq \sum y^i y^i \leq 2,
\]

\[
F^i = 0 \quad \text{elsewhere on} \quad N.
\]

The induced function \( F^i(H(x)) \) is by hypothesis of class \( C^r \) on \( M \), hence \( y^i(x) \) is of class \( C^r \) near \( P \).

Lemma 3. Let \( M \) be a differentiable manifold of class \( C^r \) provided with a Riemannian metric \( g_{ij} \) of class \( C^{r-1} \). Let \( P_0 \in M \), let \( \eta \) be a unit contravariant vector at \( P_0 \), and let \( \epsilon \), \( \epsilon > 0 \). Then there exists a function \( f \) of class \( C^r \) on \( M \) with the following properties:

1. \[ \frac{\partial f}{\partial x^i} \eta^i = 1 \quad \text{at} \quad P_0, \]
2. \[ \sup_{x \in M} \left( g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2} < 1 + \epsilon, \]
3. \[ \sup_{x \in M} |f(x)| < \epsilon. \]

Proof. Let \( (x) \) be an admissible coordinate system about \( P_0 \), with \( P_0 \) as origin, with domain \( \sum x^i x^i < 1 \), with \( g_{ij}(x) = \delta_{ij} \) at \( (x) = (0) \), and such that the components of \( \eta \) in the coordinate system \( (x) \) are \((1, 0, \cdots, 0)\). Let \( d < \epsilon \) be so small that by continuity of \( g^{ij} \) we have \[ |g^{ij}(x) - \delta^{ij}| < \epsilon/n^2 \] for \( \sum x^i x^i \leq d^2 \). Then the following is the required function:

\[
f = \left( 1 - \frac{\sum x^i x^i}{d^2} \right)^{r+1} x^1 \quad \text{for} \quad \sum x^i x^i \leq d^2,
\]

\[
f = 0 \quad \text{elsewhere on} \quad M.
\]

To show this, note first that on \( \sum x^i x^i = d^2 \) all derivatives of \( f \) up to and including those of \( r \)th order are zero, so that \( f \) is of class \( C^r \) over \( M \). Next, note that
\[
\frac{\partial f}{\partial x^1} = \left(1 - \sum \frac{x^i x^i}{d^2}\right)^r \left(1 - \sum \frac{x^i x^i}{d^2} - \frac{2(r + 1)(x^1)^2}{d^2}\right),
\]
\[
\frac{\partial f}{\partial x^a} = -\frac{2(r + 1)x^1 x^a}{d^2} \left(1 - \sum \frac{x^i x^i}{d^2}\right)^r, \quad \alpha = 2, 3, \ldots, n.
\]

It is clear that at \(P_0\) we have
\[
\frac{\partial f}{\partial x^1} = 1, \quad \frac{\partial f}{\partial x^a} = 0, \quad \frac{\partial f}{\partial x^i} \eta^i = 1.
\]

Now compute as follows:
\[
\sum \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} = \left(1 - \sum \frac{x^i x^i}{d^2}\right)^{2r} \left[\left(1 - \sum \frac{x^i x^i}{d^2} - \frac{2(r + 1)(x^1)^2}{d^2}\right)^2 + \frac{4(r + 1)^2(x^1)^2}{d^4} \sum x^a x^a\right]
\]
\[
= \left(1 - \sum \frac{x^i x^i}{d^2}\right)^{2r} \left[\left(1 - \sum \frac{x^i x^i}{d^2}\right)^2 + \frac{4(r + 1)(x^1)^2}{d^4} ((2 + r) \sum x^i x^i - d^2)\right].
\]

When \(0 \leq \sum x^i x^i \leq d^2/(2 + r)\), we have
\[
\sum \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \leq \left(1 - \frac{x^i x^i}{d^2}\right)^{2r+2} \leq 1
\]
while when \(d^2/(2 + r) \leq \sum x^i x^i \leq d^2\), we have
\[
\sum \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \leq \left(1 - \frac{x^i x^i}{d^2}\right)^{2r} \left[\left(1 - \frac{x^i x^i}{d^2}\right)^2 + \frac{4(r + 1) \sum x^i x^i}{d^4} ((2 + r) \sum x^i x^i - d^2)\right]
\]
\[
= \left(1 - \frac{x^i x^i}{d^2}\right)^{2r} \left[1 - \frac{3 + 2r}{d^2} \right] = (1 - A)^{2r}(1 - AB)^2
\]
where \(A = (\sum x^i x^i)/d^2, B = 3 + 2r\). This non-negative quantity is zero when \(A = 1\), and equal to \(((1+r)/(2+r))^{2r+2}\) when \(A = 1/(2+r)\).

Hence if we find its derivative with respect to \(A\) is zero at a unique value \(\bar{A}\) between \(A = 1/(2+r)\) and \(A = 1\), this locates its maximum.
either at $\overline{A}$ or at $A = 1/(1 + 2r)$. But

$$
\frac{d}{dA} ((1 - A)^{2r}(1 - AB)^2)
= 2(1 - A)^{2r-1}(1 - AB)(- B + AB - r + rAB)
$$

and since $1 - AB < 0$ for $1/(2 + r) < A < 1$, the only root in this $A$-interval is $A = (B + r)/(B + rB)$. For this value of $A$, we find

$$(1 - A)^{2r}(1 - AB)^2 = 4/(1 + 3/2r)^2 < 1.$$

Thus

$$
\sum \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^i} \leq 1 \quad \text{for} \quad 0 \leq \sum x^i x^i \leq d^2.
$$

Now by our choice of $d$

$$
g_{ij}(x) < \delta_{ij} + \epsilon/n^2 \quad \text{for} \quad 0 \leq \sum x^i x^i \leq d^2
$$

so that

$$
g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^i} < \sum \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^i} + \epsilon \leq 1 + \epsilon.
$$

Hence all over $M$

$$
\left( g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^i} \right)^{1/2} < 1 + \epsilon.
$$

As for $\sup |f(x)|$, note that $f = 0$ at $(x) = (0)$ and on $\sum x^i x^i = d^2$, and that $f$ is an odd function of $x^1$, so that $\max |f(x)|$ in $0 \leq \sum x^i x^i \leq d^2$ is the same as $\max f(x)$ and occurs at a critical point of $f$ not on $\sum x^i x^i = d^2$. From the form of $\partial f/\partial x^i$, it is seen that the critical point wanted is

$$
x^\alpha = 0, \quad \alpha = 2, 3, \ldots, n,
x^1 = d/(1 + 2r)^{1/2}.
$$

Hence

$$
\max |f(x)| = \left( 1 - \frac{1}{1 + 2r} \right)^r \frac{d}{(1 + 2r)^{1/2}}
= \left( \frac{2r}{1 + 2r} \right)^r \frac{d}{(1 + 2r)^{1/2}} < d < e.
$$