In a recent paper of Wintner [1], an extension is made of a classical
theorem on the Legendre transformation of a convex function (for
details of the proof, see [3], and for related results, cf. [4; 5]). His
assumptions are that the function be strictly convex and of class C'1.
Here we shall prove a more general result which eliminates both of
these restrictions and shows that, in a sense, they are dual to one
another. As two applications we mention a result of Kamke [2] on
the Clairaut differential equation and a theorem on parallel curves
(cf. [3, pp. 481–482]).

Let \( y(x) \) be a convex bounded function of \( x \) on the interval \((a, b)\).
Then \( y \) has monotone nondecreasing right- and left-hand derivates
\( y'_-(x) \leq y'_+(x) \) in \((a, b)\). There will, in general, be two exceptional sets
to consider in reference to these derivates, the set \( j \) consisting of
points \( x \) where \( y'_-(x) \not= y'_+(x) \) and the set \( k \) of the closures of the
\( x \)-intervals where \( y'_+(x) \) (or \( y'_-(x) \)) is constant. \( j \) is an at most de-
numerable set of points while \( k \) is an at most denumerable set of
closed intervals. Let \( K \) be the set of closures of intervals of \( y' \)-values
satisfying \( y'_-(x) < y' < y'_+(x) \) for some \( x \) on \( j \), and \( J \) the set of \( y' \)-values
for which \( y' = y'(x) \) for some \( x \) on \( k \). Furthermore, let \( A = y'_+(a) \),
\( B = y'_-(b) \).

We now consider the correspondence \( X = y'(x) \). This obviously
assigns to every \( x \) not in \( j \) a unique \( X \) not in \( K \) and to every \( X \) not in
\( J \) a unique \( x \) not in \( k \). Thus \( X = y'(x) \) is a unique (monotone and con-
tinuous) correspondence of \((a, b) - (j + k) \) onto \((A, B) - (J + K) \). In
order to extend the domain of definition of this correspondence, put
\( \bar{x} = \bar{x}(X) = 1.\text{u.b.} \) \( T(X) \), where \( T(X) \) is the set of those \( x \)-values at
which \( y'_-(x) \leq X \). \( \bar{x}(X) \) is in \( T(X) \) and \( \bar{x}(X) \) is continuous from the
right.

We now define \( Y = Y(X) \) as

\[
Y = \bar{x}(X) \cdot X - y(\bar{x}(X)).
\]

Letting \( ' \) after \( Y \) denote differentiation with respect to \( X \) we have the following

**Theorem.** \( Y'_+ \) and \( Y'_- \) exist everywhere and are nondecreasing with
\( Y'_- \leq Y'_+ \) and \( Y'_+(X) = \bar{x}(X) \). Furthermore, \( Y'_- \not= Y'_+ \) only on \( J \) and \( Y'_+ \)
can constant only on intervals of \( K \).

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In proving this theorem we make use of the following extension of the mean value theorem:

**Lemma.** If \( f(t) \) has in \((a, b)\) a right- and left-hand derivative at every point and if \( t' < t'' \) are points of \((a, b)\), then there exists a \( t^* \) such that \( t' < t^* < t'' \) and \((t'' - t')f^-((t^*)) \leq f(t'') - f(t') \leq (t'' - t')f^+(t^*) \) or \((t'' - t')f^+(t^*) \leq f(t'') - f(t') \leq (t'' - t')f^-(t^*)\).

To see this set
\[
g(t) = f(t) - (t - t') \frac{f(t'') - f(t')}{t'' - t'}.
\]
g\((t)\) has right- and left-hand derivates in \((t', t'')\) and is certainly continuous on the closure of this interval. Furthermore \( g(t'') = g(t') = f(t') \) and therefore \( g(t) \) assumes a maximum or a minimum in the interior of the interval, say at \( t = t^* \). Then \( g^+(t^*) \leq 0, g^-(t) \geq 0 \) or \( g^+(t^*) \leq 0, g^-(t^*) \geq 0 \) according as \( t^* \) is a maximum or a minimum. This is the desired result.

We shall now show that \( Y_+(x) \) exists everywhere and is equal to \( \bar{x}(x) \). Let \( x_1 \) be greater than \( x \) and \( \Delta x = x_1 - x, \Delta y = \bar{x}(x_1) - \bar{x}(x), \) etc. Then
\[
\Delta x = x_1 \Delta x + X \Delta \bar{x} - \Delta y(\bar{x}).
\]
By the previous lemma, there exists an \( x^* \) such that \( \bar{x} < x^* < x_1 \) (unless \( \bar{x} = \bar{x}_1 \) in which case \( \Delta y = \bar{x}(x) \)) and
\[
(\Delta \bar{x})y^-(x^*) \leq \Delta y \leq (\Delta \bar{x})y^+(x^*),
\]
so that
\[
\Delta Y \leq \bar{x}_1 \Delta x + X(\Delta \bar{x}) - (\Delta \bar{x})y^+(x^*).
\]
But \( X \leq y^-(x^*) \leq X_1 \), hence
\[
(2) \quad \Delta Y \leq \bar{x}_1 \Delta x.
\]
Similarly,
\[
\Delta Y = x_1 \Delta \bar{x} + x \Delta X - \Delta y(\bar{x})
\]
and, since \( x^* \) is strictly between \( \bar{x} \) and \( \bar{x}_1 \) (unless \( \bar{x} = \bar{x}_1 \) in which case \( \Delta y = \bar{x}(x) \)), \( X \leq y^-(x^*) \leq X_1 \). Thus
\[
(3) \quad \Delta Y \geq \bar{x} \Delta X.
\]
We therefore have, by (2) and (3),
\[
\bar{x} \leq \Delta Y/\Delta X \leq \bar{x}_1.
\]
Letting $X_1$ tend to $X$ we get, since $\bar{x}$ is continuous from the right, $Y^+(X) = \bar{x}(X)$.

Since $\bar{x}$ is a nondecreasing function of $X$, $Y^+_+$ is nondecreasing. Furthermore, $Y^-(X) = Y^+_+(X-0) = \bar{x}(X-0)$, $Y^-(X) \leq Y^+_+(X)$, and $Y^-(X)$ is nondecreasing. The points where $Y^-(X) \neq Y^+_+(X)$ are those where $\bar{x}(X-0) \neq \bar{x}(X)$ or where $y^-(x)$ is constant on some interval to the left of $\bar{x}$, thus belonging to $J$. Similarly $Y^+_+ = \bar{x}$ is constant only on intervals of $K$.

$Y$ is thus a convex function of $X$. We can, therefore, apply the preceding transformation (1) on $Y = Y(X)$. We thus put $x = Y'(X)$ on $[A, B]-(J+K)$ and $\bar{x} = \bar{x}(x)$ = l.u.b. $t(x)$ where $t(x)$ is the set of $X$ such that $Y^-(X) \leq x$.

Finally, let $y^* = y^*(x)$ be defined by

$$y^* = \bar{x}(x) \cdot x - Y[\bar{x}(x)] = \bar{x}(x) \cdot x - \bar{x}[\bar{x}(x)]\bar{x} + y(\bar{x}[\bar{x}(x)]).$$

Now, by the definitions of the functions $\bar{x}(X)$ and $\bar{x}(x)$, $\bar{x}[\bar{x}(x_0)]$ is the largest $x$ such that $y^-(x) \leq y^-(x_0)$. For $x$ not in $k$, $\bar{x}[\bar{x}(x)] = x$, in which case

$$y^* = \bar{x} \cdot x - x \cdot \bar{x} + y(x) = y.$$

For $x$ in an interval of $k$, $y(x)$ is linear with the constant slope $\bar{x}$ and assumes the value $y(\bar{x}[\bar{x}(x)])$ at $x = \bar{x}$, and so the last part of (4) shows that $y^* = y$.

Hence $y^* = y$ for all $x$ in $[a, b]$. Thus the transformation (1) treated above is involutory with

$$Y = Y(X) = \bar{x}X - y(\bar{x}), \quad Y^+(X) = \bar{x}$$

and

$$y = y(x) = \bar{x}x - Y(\bar{x}), \quad y'(x) = \bar{x}.$$

If $y$ has a derivative everywhere, then $j$ is empty and therefore $K$ is empty. Thus if $y$ is differentiable everywhere, $Y$ is strictly convex and vice versa. Similarly, if $y$ is strictly convex, the $Y$ has a derivative everywhere and vice versa. Finally, if $y$ or $Y$ are strictly convex and have derivatives everywhere, or if both are strictly convex or both are everywhere differentiable, then both are strictly convex and everywhere differentiable.

As a first application, we mention the following theorem:

The (outside) parallel curves of a convex curve are of class $C^1$ (at least).

In effect, rolling a circle outside the curve increases the support
function which means adding a strictly convex term to the Legendre transform (for suitable choice of coordinates).

As a second application, we mention the result of Kamke [2] on the Clairaut differential equation, namely, the existence of a non-linear solution of $y = y'x + f(y')$ when $f$ is strictly convex. Here, the Legendre transform of $f$ provides a solution if it is differentiable, which it must be in virtue of the strict convexity of $f$.

REFERENCES