HAAR MEASURE AND THE SEMIGROUP OF MEASURES ON A COMPACT GROUP

J. G. WENDEL

1. Introduction. Crucial results in the theory of a compact topological semigroup $S$ state that $S$ must possess idempotents, and that if $S$ contains an identity and is not a group then at least one additional idempotent occurs [6; 7; 8]. These facts were pointed out to the author by R. J. Koch, whom it is here a pleasure to thank for many stimulating and instructive conversations.

An interesting example is furnished by the set $S$ of (normalized nonnegative regular) measures on a compact group $G$; $S$ becomes a compact Hausdorff semigroup with identity under natural definitions of multiplication and topology, and $S$ is not a group if $G$ has more than one element. Then $S$ has additional idempotents; what are they? If the existence of Haar measures is assumed it is easy to show that the Haar measure associated with any compact subgroup of $G$ determines an idempotent. The converse is more interesting, and is our main concern here: the existence of idempotents in $S$ implies the existence of Haar measure on $G$. (At this writing we are unable to apply the method to general locally compact groups.) As a secondary result we show that $G$ is determined by $S$.

2. Definitions and preliminary results. Let $G$ be a compact group and let $S$ be the set of countably additive nonnegative regular set functions $\mu$ on the Borel sets of $G$ with $\mu(G) = 1$. Let $C(G)$ be the Banach space of real continuous functions on $G$, and recall the 1-1 correspondence between measures $\mu \in S$ and continuous linear functionals $\phi$ on $C(G)$, with the properties $f(x) \geq 0$ implies $\phi(f) \geq 0$ and $\phi(1) = 1$, that is given by the equation $\phi(f) = \int f(x)d\mu(x)$. For later reference we note the formula (cf. [3, Theorem 56. E])

$$
\mu(U) = \sup \left\{ \int f(x)d\mu(x) \mid f \in C(G), \right. \\
\left. 0 \leq f(x) \leq 1, f(x) = 0 \text{ for } x \notin U \right\}
$$

valid for all open sets $U$ in $G$.

We give $S$ the weak* topology for functionals, so that $\mu_n \to \mu$ means

Presented to the Society, February 27, 1954; received by the editors January 11, 1954 and, in revised form, March 15, 1954.

923
\[ \int f(x) d\mu_a(x) \rightarrow \int f(x) d\mu(x), \quad f \in C(G). \]  

S is compact in this topology, by virtue of the weak* compactness of the unit sphere in the conjugate space of \( C(G) \) and the fact that the restrictions on \( \phi ' \)'s imposed above determine a closed subset of the unit sphere.

As in [2] we define multiplication of \( \mu, \nu \in S \) by

\[ (2) \quad \int f(x) d(\mu \nu)(x) = \int \int f(yz) d\mu(y) d\nu(z), \quad f \in C(G); \]

it is easy to see that the multiplication is associative and continuous on \( S \). Thus \( S \) is a compact topological semigroup.

Now for any element \( x \in G \), we define the element \( x' \in S \) as the point mass at \( x \), i.e. \( x'(E) = 1 \) if \( x \in E \), 0 if \( x \not\in E \). The corresponding functional sends the function \( f \) into the number \( f(x) \), and the group element \( xy \) goes over into the measure \( (xy)' = x'y' \). Therefore the mapping \( x \rightarrow x' \) of \( G \) into \( S \) is a homeomorphic isomorphism, so that henceforth we may regard \( G \) as embedded in \( S \) and omit primes. Clearly the identity \( e \) of \( G \) is also the identity for \( S \); in fact, making use of (2) it is easy to see that for \( x \in G \)

\[ (\mu x)(E) = \mu(Ex^{-1}), \quad (x\mu)(E) = \mu(x^{-1}E), \quad \text{all } E. \]

The semigroup \( S \) has two special structural features not shared by all compact semigroups with identity, which are: the possibility of forming convex linear combinations of elements of \( S \), and the presence of a natural antiautomorphic involution in \( S \); it is doubtful whether we have made maximal use of these properties, but at any rate they turn out to be very useful.

More explicitly, for \( \mu, \nu \in S \) and \( t \) real, \( 0 \leq t \leq 1 \), \( t\mu + (1-t)\nu \) is again in \( S \). This remark enables us to exhibit elements of \( S \) which do not possess inverses if \( G \) has more than one element, so that \( S \) is not a group. In fact, for \( x, y \in G \) the measure \( (1/2)x + (1/2)y \) has no inverse; this is a consequence of the following lemma whose proof we defer to the next section.

**Lemma 1.** A necessary and sufficient condition that \( \mu \in S \) have an inverse is that \( \mu \) is a point mass, \( \mu \in G \).

From this it follows easily that \( S \) determines \( G \), for in order to reconstruct \( G \) from \( S \) we have only to pick out the elements with inverses.

The involution \( \mu \rightarrow \mu^* \) is defined by \( \mu^*(E) = \mu(E^{-1}) \); the assertion \( \mu^{**} = \mu \) is trivial, and the fact that \( (\mu \nu)^* = \nu^* \mu^* \) follows easily from \( \int f(x) d\mu^*(x) = \int f(x^{-1}) d\mu(x) \) and equation (2). A deeper fact about the involution is contained in
Lemma 2. If \( \mu \) is idempotent, then \( \mu^* = \mu \).

Again the proof is postponed to the next section.

3. The idempotents of \( S \). Let \( H \) be a compact subgroup of \( G \) and suppose that \( H \) has Haar measure \( \mu \). We extend the definition of \( \mu \) to all the Borel sets of \( G \) by putting \( \mu(E) = \mu(E \cap H) \), and show that \( \mu \) is idempotent. This follows from the more general statement that if \( \nu \in S \) is carried on \( H \) (this means that \( \nu(H) = 1 \)), then \( \mu \) annihilates \( \nu \), \( \nu^* = \mu = \mu \). In fact, to show that \( \nu^* = \mu \) we have

\[
\int_{x} f(x) d(\nu \mu)(x) = \int_{H} \int_{H} f(yz) d\nu(y) d\mu(z) = \int_{H} \int_{H} f(y) d\mu(y) d\nu(z) = \left\{ \int_{H} f(y) d\mu(y) \right\} \nu(H) = \int_{x} f(x) d\mu(x), \quad f \in C(G),
\]

and similarly \( \nu \mu = \mu \).

We remark two important special cases: if \( H = \{ e \} \) then \( \mu = e \), and if \( H = G \) then \( \mu = \text{Haar measure of } G = \text{annihilator of } S \). Of course here we are assuming that \( G \) has Haar measure.

Conversely, without assuming the existence of Haar measure, we are going to prove

Theorem 1. Let \( \mu \) be an idempotent in \( S \). There exists a unique compact subgroup \( H \) in \( G \) such that \( \mu \) is the Haar measure of \( H \), extended as above to all of \( G \).

Before proving the theorem we need some lemmas, as follows.

Lemma 3. For any \( \mu \in S \) there is a unique closed set \( A \), the carrier of \( \mu \), such that \( \mu(A) = 1 \) and \( \mu(U) > 0 \) for any nonempty relatively open subset \( U \) of \( A \).

Proof. The uniqueness of \( A \) is easy, for if \( A' \) were a second closed set with the specified properties we could choose the notation so that \( U = A' - A \) would be nonempty and write

\[
1 = \mu(G) = \mu(A) + \mu(A' - A) = 1 + \mu(A' - A) > 1,
\]

a contradiction.

For the proof of existence let \( \mathcal{F} \) be the family of closed sets \( F \) with \( \mu(F) = 1 \). \( G \in \mathcal{F} \); hence \( \mathcal{F} \) is not empty. For \( F_1, F_2, \ldots, F_n \in \mathcal{F} \) we find that \( \cap F_i \) again belongs to \( \mathcal{F} \), since the complements of the \( F_i \) are \( \mu \)-nullsets. Thus \( \mathcal{F} \) has the finite intersection property; let \( A \) be the closed nonempty set of points common to all the \( F_i \)'s. Let \( U \) be an open set containing \( A \). Then \( U \) contains some finite intersection of \( F_i \)'s. Therefore \( \mu(U) = 1 \). Then the regularity of \( \mu \) shows that \( \mu(A) = 1 \). No closed proper subset of \( A \) belongs to \( \mathcal{F} \), i.e. has \( \mu \)-measure equal to 1, and this observation completes the proof of the lemma.

Lemma 4. Let \( A \) and \( B \) be the carriers of \( \mu \) and \( \nu \). Then \( AB = \{ xy \mid x \in A, y \in B \} \) is the carrier of \( \mu \nu \).
We shall first show that the desired relation follows from

\[(\nu \nu)(VW) \geq \mu(V)\nu(W), \quad V \text{ and } W \text{ open in } G.\]

Let \(U\) be an open set containing \(AB\); by the continuity of multiplication we may find open \(V\) and \(W\) containing \(A\) and \(B\) such that \(VW\) is contained in \(U\). Then

\[(\nu \nu)(U) \geq (\nu \nu)(VW) \geq \mu(V)\nu(W) \geq \mu(A)\nu(B) = 1;\]

hence by the regularity of \(\nu \nu\), \((\nu \nu)(AB) = 1.\) Similarly, if \(U\) is open in \(AB\) it follows easily that \((\nu \nu)(U) > 0.\) Therefore \(AB,\) which is closed by a known theorem, has the required properties.

It remains to establish (3), which we shall do by invoking (1). Let \(g\) and \(h\) be continuous maps of \(G\) to \([0, 1]\) vanishing outside of \(V\) and \(W\) respectively. Define a real function \(f\) on \(G\) by \(f(z) = \max_x g(x)h(x^{-1}z).\) Clearly \(f\) is continuous, \(f\) sends \(G\) to \([0, 1]\), and \(f\) vanishes outside of \(VW.\) Moreover, \(f(xy) \geq g(x)h(y).\) Thus \(\int f(z)d(\nu \nu)(z) = \int \int f(xy)d\mu(x)d\nu(y) \geq \int g(x)d\mu(x)\int h(y)d\nu(y);\) taking suprema over \(g\) and \(h\) and applying (1) then yields (3).

**Proof of Lemma 1.** If \(\mu \nu = e\) and \(A, B\) are the carriers of \(\mu, \nu,\) then \(AB = \{e\}.\) Hence \(A\) and \(B\) are one-point subsets of \(G,\) as had to be shown.

**Lemma 5.** If \(\mu\) is idempotent and \(H\) its carrier, then \(H\) is a compact subgroup of \(G.\)

**Proof.** By Lemma 4, \(H^2 = H.\) Therefore by a known theorem \([1; 6]\) \(H\) is a compact subgroup.

**Proof of Theorem 1.** Let \(H\) be the carrier of \(\mu.\) By Lemma 3 every open set of \(H\) has positive \(\mu\)-measure and, by Lemma 5, \(H\) is a compact subgroup of \(G.\) To show that \(\mu\) is translation-invariant on \(H\) we must show that for any \(f \in C(H)\) the expression

\[g(y) = \int f(xy)d\mu(x)\]

is constant for \(y \in H.\) The function \(g\) is continuous and attains its maximum at a point which, after replacing \(f\) by a suitable translate, we may take to be \(y = e.\) Then from (2) we have

\[g(e) = \int f(x)d\mu(x) = \int \int f(yz)d\mu(y)d\mu(z) = \int g(z)d\mu(z) \leq \int g(e)d\mu(z) = g(e).\]
Therefore \( g \) is identically \( g(e) \), and since the uniqueness of \( H \) is trivial Theorem 1 is proved.

**Proof of Lemma 2.** This is an easy consequence of Theorem 1 and the known fact applied to \( H \) that on a compact group Haar measure is invariant under inverse.

4. **The existence of Haar measure.** Now we want to apply the foregoing considerations to give a new proof of the existence of Haar measure for compact groups. Let \( G \neq \{ e \} \); then, as we have seen, \( S \) is a compact semigroup with identity, which is not a group. There is then another idempotent in \( S \), and by Theorem 1 some nontrivial subgroup \( H \) of \( G \) has Haar measure. The idea of the proof is to show first that there is a largest such subgroup \( H \), which is necessarily normal in \( G \); then if \( H \) is not already \( G \) we shall show that we have a contradiction.

We begin by partially ordering the compact subgroups \( H \) of \( G \) which have Haar measure under inclusion. The results of the previous section show that this is equivalent to partially-ordering the idempotents of \( S \) by annihilation: \( \mu \leq \nu \) means \( \mu \nu = \nu \mu = \nu \).

**Lemma 6.** The set of idempotents is a directed set in the above ordering.

**Proof.** Let \( \mu, \nu \) be distinct idempotents of \( S \). Form \( \sigma = \mu \nu \) and let \( T \) be the closure in \( S \) of the set of powers of \( \sigma \). It is clear that \( T \) is a compact subsemigroup of \( S \). Since \( \mu \sigma^n = \sigma^n \nu = \sigma^n \) we have \( \mu \tau = \tau \nu = \tau \) for \( \tau \in T \). (If \( G \) (and therefore \( S \)) is abelian, then trivially \( T = \{ \sigma \} = \{ \mu \nu \} \).) By the fundamental theorem there is an idempotent \( \tau \) in \( T \). Applying the involution and Lemma 2 we find that \( \tau \) annihilates both \( \mu \) and \( \nu \). That is, \( \tau \geq \mu, \tau \geq \nu \). (Recent results of Koch [4] show that \( \tau \) is unique; from this it follows easily that the carrier of \( \tau \) is the closed subgroup generated by the carriers of \( \mu \) and \( \nu \).)

**Lemma 7.** There is a greatest idempotent in \( S \).

**Proof.** The set of idempotents is defined by the relation \( \mu^2 = \mu \) and therefore is compact. For each \( \mu \) let \( Q(\mu) = \{ \nu \mid \nu \geq \mu \} \). These sets are closed, since they are defined by relations \( \nu^2 = \nu = \nu \mu = \mu \nu \). By Lemma 6 the family of \( Q(\mu) \)'s has the finite intersection property, and hence there is a point common to all of them. Another application of Lemma 6 shows that there is precisely one such point, and this is the desired greatest idempotent.

**Lemma 8.** Let \( H \) be the carrier of the greatest idempotent \( \mu \). Then \( H \) is a nontrivial compact normal subgroup of \( G \).

**Proof.** \( H \) is a compact subgroup of Theorem 1: \( H \neq \{ e \} \) since \( e \) is
certainly not the greatest idempotent. Finally we show that $H$ is normal. Let $x \in G$. Then the measure $x^{-1} \mu x$ is idempotent, and its carrier is $x^{-1} H x$, by Lemma 4. We must have $x^{-1} \mu x = \mu$ with the consequent $x^{-1} H x = H$, since otherwise $\mu > x^{-1} \mu x$, from which it would easily follow that $x \mu x^{-1} > \mu$, a contradiction.

**Theorem 2.** $G$ has Haar measure.

**Proof.** If $H$ of Lemma 8 is $G$ we are finished. But if $H \neq G$ we can construct an idempotent greater than $\mu$ in the following way. As shown in the proof of Lemma 8, $\mu$ commutes with each element of $G$. Then also $\mu$ commutes with each $\nu \in S$, for \[ \int \int f(xy) d\mu(x) d\nu(y) = \int \int (yxy^{-1}) d\mu(y) d\nu(x) = \int \int f(xy) d\mu(y) d\nu(x) = \int \int f(xy) d\nu(y) d\mu(x). \]

Let $S' = S \mu$, a compact semigroup with identity $\mu$. $S'$ is not a group, for if $x \in H$ the element $(1/2)x + (1/2)/x$ can have no inverse in $S' = S \mu$, by an argument with carriers similar to that used in proving Lemma 1. Then $S'$ contains an idempotent $\nu = \nu \mu \neq \mu$, and $\nu \mu = \nu \mu \neq \mu$. This contradiction completes the proof.

It may be worthwhile to sketch the mechanism underlying this proof. One might attempt to prove Theorem 2 as follows. If $H \neq G$, form the compact group $G' = G/H$, and find therein a nontrivial subgroup $K'$ having Haar measure; by a modification of a device used in [3; 5] one can then construct Haar measure on $K$, the antecedent of $K'$ in the natural map of $G$ upon $G'$; $K$ properly contains $H$, a contradiction. The precise connection between these remarks and the given proof is found in

**Theorem 3.** $S' = S \mu$ may be identified with the semigroup $S(G')$ of measures on $G' = G/H$. (This does not depend on maximality of $H$, but only on its normality in $G$.)

**Proof.** Let $\nu' \in S(G')$ and $f \in C(G)$. Form $F(x) = \int f(xy) d\mu(y) = \int \int f(xy) d\mu(y) d\mu(z)$. $F$ is continuous and is constant on cosets $x'$ of $H$ and therefore defines an element of $C(G')$. Form $\int F(x) d\nu'(x')$, where $x$ is any element of $x'$. This defines a functional of the appropriate kind on $C(G)$, and so may be written in the form $\int f(x) d\nu(x) = \int \int f(xz) d\nu(x) d\mu(z)$ with $\nu \in S$. Hence $\nu = \nu \mu \in S \mu = S'$. It is routine to verify that the mapping $\nu' \mapsto \nu \mu$ is a continuous (and therefore homeomorphic) isomorphism of $S(G')$ onto $S'$; we suppress further details.

**Bibliography**


Louisiana State University

**ON THE SPECTRA OF COMMUTATORS**

C. R. PUTNAM

1. The following theorems will be proved:

(*) Let $A$ and $B$ denote bounded operators in a Hilbert space and suppose that the commutator

(1) \[ C = AB - BA \]

satisfies the commutation relation

(2) \[ AC = CA. \]

Then there exists a sequence of bounded operators $C_1, C_2, \cdots$ such that

(3) \[ \|C - C_n\| \to 0 \quad \text{and} \quad \text{sp} (C_n) \to 0 \quad \text{as} \quad n \to \infty. \]

(**) If, in addition to the assumption (2) of (*), the relation

(4) \[ BC = CB \]

also holds, then the assertion (3) can be improved to

(5) \[ \text{sp} (C) \text{ consists of } 0 \text{ alone}. \]

For the terminology see, e.g., [3, pp. 9–10]. It is understood that \sp (D) denotes the set of points in the spectrum of an operator $D$. Thus, the relation \sp (C_n) \to 0 means that for any $\epsilon > 0$ there is a number $N_\epsilon$ such that the spectrum of $C_n$ is contained in the disk $|\lambda| < \epsilon$ whenever $n > N_\epsilon$.

As a consequence of (**) one obtains the

**COROLARY.** If $A$ is a bounded operator in a Hilbert space which com-

Received by the editors March 23, 1954.

1 This work was supported in part by the National Science Foundation research grant NSF-G481.