A NOTE ON DENDRITES AND TREES

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By a *tree* we mean a continuum (=compact connected Hausdorff space) in which every pair of distinct points is separated by a third point. It is well known [4] that a dendrite is simply a metric tree. In this note we give a new characterization of dendrites and trees in terms of partially ordered spaces.

Recall from [3] that a POTS (=partially ordered topological space) is a topological space X endowed with a partial order, $\leqq$, which is semicontinuous in the sense that $L(x)=\{a:a\leqq x\}$ and $M(x)=\{a:x\leqq a\}$ are closed sets for each $x\in X$. A partial order is *order dense* if $x<y$ implies there is a z such that $x<z<y$. A *chain* of a partially ordered set is a subset which is linear with respect to the partial order. An element in a partially ordered set is *minimal* (*maximal*) if it has no proper predecessor (successor).

Suppose that K is a locally connected continuum and $e\in K$. Define a relation, $\leqq$, in K by $x\leqq y$ if, and only if, $x=e$, or $x=y$, or x separates e and y in K. From [3] we have

LEMMA 1. *The relation $\leqq$ is a semicontinuous partial order.*

We shall require two more results from [3]. Lemma 2 is originally due to Wallace [2].

LEMMA 2. *A compact POTS contains a maximal element.*

LEMMA 3. *A compact order dense POTS is connected if its set of minimal elements is connected.*

LEMMA 4. *A tree is locally connected.*

PROOF. Let X be a tree; by [1, p. 140] X is regular in the sense that if $x\in U\subset X$, U open, then there is a neighborhood V of x, with $\overline{V}\subset U$ and such that the frontier of V is finite. By [4, p. 19] X is locally connected. Since the above arguments are valid in our situation, the proof is complete.

We may now prove our chief result.

THEOREM 1. *Let X be a compact Hausdorff space. A necessary and sufficient condition that X be a tree is that X admit a partial order, $\leqq$, satisfying*

(i) *$\leqq$ is semicontinuous,*

Presented to the Society, June 19, 1954; received by the editors April 23, 1954.

(ii) $\leqq$ *is order dense,*
(iii) *for* $x \in X$, $y \in X$, *it follows that* $L(x) \cap L(y)$ *is a non-null chain,*
(iv) $M(x) - x$ *is an open set, for each* $x \in X$.

PROOF. Let X be a tree; choose $e \in X$ and let X have the semi-continuous partial order of Lemma 1. To establish (ii), let $x < y$ and suppose z separates x and y, i.e.,

$$X - z = C \cup D, \qquad C \mid D, \qquad x \in C, \qquad y \in D.$$

If $x = e$, then, by definition, $x < z < y$. Otherwise, we have

$$X - x = A \cup B, \qquad A \mid B, \qquad e \in A, \qquad y \in B.$$

It is easy to see that $x \in C$ and $y \in B$ imply $A \subset C$. Therefore $e \in C$, so that $z < y$. Likewise, $z \in B$, so that $x < z$. To prove (iii), we first note that by definition of the partial order, $e \in L(x) \cap L(y)$ for any x and y in X. It remains to show that $L(x)$ is a chain. If $e < p_1, p_2 < x$, we have

$$X - p_i = A_i \cup B_i, \qquad A_i \mid B_i \qquad (i = 1, 2),$$
$$x \in B_1 \cap B_2, \qquad e \in A_1 \cap A_2.$$

It is clear that $p_1 \in \overline{A}_2$ implies $p_1 \leqq p_2$ and $p_1 \in \overline{B}_2$ implies $p_2 \leqq p_1$. To establish (iv), we note simply that $M(x) - x = X - x$ if $x = e$, and if $x \neq e$, then

$$M(x) - x = \bigcup \{B_\alpha : X - x = A_\alpha \cup B_\alpha, A_\alpha \mid B_\alpha, e \in A_\alpha\}.$$

In either case, $M(x) - x$ is open. Thus the condition is necessary.

Suppose now that X admits a partial order satisfying (i)–(iv). By Lemma 3, X is connected and hence is a continuum. If $x < y$, then by (ii) there is $z \in X$ such that $x < z < y$. By (i) and (iv), $M(z)$ is closed and $M(z) - z$ is open, so that z separates x and y. If x and y are not comparable, then by (iii) and Lemma 2, there exists $z = \max\ (L(x) \cap L(y))$. Choose t such that $z < t < x$. Then $x \in M(t) - t$ and $y \in X - M(t)$, whence t separates x and y. This completes the proof.

By Theorem 1 and Lemma 4, we have at once:

COROLLARY. *Let X be a compactum. A necessary and sufficient condition that X be a dendrite is that X admit a partial order satisfying* (i)–(iv) *of Theorem* 1.

BIBLIOGRAPHY

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CORRECTION TO "UNIQUENESS OF THE PROJECTIVE PLANE WITH 57 POINTS"

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Professor Gunter Pickert has pointed out to me that there is an error in my paper [Proc. Amer. Math. Soc. vol. 4 (1953) pp. 912–916]. The equation on page 915 which reads $12+3s+2t=24$ should read $12+3s+2t+u=24$. This invalidates my conclusion $u=0$ from which I deduce $U=0$, which is necessary for the rest of the paper. I give here a new proof that $U=0$.

We must show that in a 57 point plane a line containing points ACC is not possible, or that $U=0$. The proof is by showing that if there is a line ACC we reach a contradiction. We may take ACC as $A_1C_4C_6$ by numbering points appropriately. We now reletter by the following rule:

$$\begin{array}{ccccccccc} A_1 & A_2 & A_3 & A_4 & B_1 & B_2 & B_3 & C_4 & C_6 \\ A_1 & B_1 & B_5 & A_0 & B_4 & B_2 & A_2 & B_3 & B_6 \end{array}$$

and obtain the following configuration:

$$\begin{array}{cccc} A_0 & B_1 & B_2 & B_3 \\ A_0 & B_4 & B_5 & B_6 \\ A_0 & A_1 & A_2 & \\ A_1 & B_1 & B_4 & \\ A_1 & B_2 & B_5 & \\ A_1 & B_3 & B_6 & \\ A_2 & B_1 & B_5 & \\ A_2 & B_2 & B_6 & \\ A_2 & B_3 & B_4 & \end{array}$$