A REMARK ON A NORM INEQUALITY FOR SQUARE MATRICES

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In a recent paper W. Gautschi [1] obtained the following inequality for a non-nilpotent $n$-square matrix $A$:

\[(1) \left( \sum_{j=1}^{n} |\lambda_j|^{2p} \right)^{1/2} \leq \|A^p\| \leq c_0 \rho^{k-1} \left( \sum_{j=1}^{n} |\lambda_j|^{2p} \right)^{1/2}\]

where $\|A\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$, $\rho$ is an integer, $c_0$ a constant depending only on $A$, and $k$ the maximum multiplicity of any characteristic root of $A$. By a reduction to Jordan canonical form an inequality of type (1) may be obtained for any convergent matrix power series $g(A)$, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n$ real, which in the case $g(z) = z^p$ implies (1).

**Theorem.** Let $g(z)$ have radius of convergence $\rho > 0$. Let $A$ be an $n$-square matrix whose characteristic roots satisfy $|\lambda_i| < \rho$. Let the multiplicity of $\lambda_i$ be $k_i$, $i = 1, \ldots, \alpha$, and $k = \max k_i$. Then if $A$ is not nilpotent there exists a constant $c$ depending only on $A$ such that

\[(\sum_{j=1}^{n} |g(\lambda_j)|^2)^{1/2} \leq \|g(A)\| \leq c \left( \sum_{i=1}^{\alpha} \sum_{s=0}^{k_i-1} \frac{(k_i - s)}{(s!)^2} |g^{(s)}(\lambda_i)|^2 \right)^{1/2} .\]

**Proof.** A very slight extension of Gautschi's argument yields the lower bound. For by Schur's theorem [2] there exists a unitary matrix $U$ reducing $A$ to triangular form $D$ with characteristic roots along the main diagonal. Then

\[U A U^* = D \quad \text{and} \quad U U^* = I\]

imply

\[\|g(A)\|^2 = \text{tr} (g(A) g^*(A)) = \text{tr} (g(A) g(A^*)) \]
\[= \text{tr} (U g(A) g(A^*) U^*) = \text{tr} (U g(A) U^* g(A^*) U^*) \]
\[= \text{tr} (g(U A U^*) g(U A^* U^*)) = \text{tr} (g(D) g(D^*)) \]
\[= \text{tr} (g(D) g^*(D)) \geq \sum_{j=1}^{n} |g(\lambda_j)|^2 \]

since $g(D)$ is a triangular matrix with $g(\lambda_j)$ along the main diagonal. To obtain the upper bound let $T$ be such a nonsingular matrix that $T A T^{-1} = \text{diag} (E_1, \ldots, E_a) = \text{diag} E_j$; $E_j = \lambda_j I_j + U_j$, $I_j$ the $k_j$-square matrix.

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identity matrix and $U_j$ the $k_j$-square matrix with 1 along the super-diagonal and 0 elsewhere. Then setting $c = \|T\| \|T^{-1}\|$, we have

$$\|g(A)\| = \|T^{-1}Tg(A)T^{-1}T\| = \|T^{-1}g(TAT^{-1})T\| \leq c\|g(\text{diag } E_j)\|$$

$$= c\|\text{diag } g(E_j)\|.$$ 

An elementary calculation shows (S. Lefschetz [3])

$$g(E_i) = g(\lambda_i I_i + U_i) = \begin{bmatrix}
  g(\lambda_i) & g^{(1)}(\lambda_i) & \cdots & g^{(k_j-1)}(\lambda_i)/(k_j - 1)!
  
  \vdots & \vdots & \ddots & \vdots
  
  \vdots & \vdots & \ddots & \vdots
  
  0 & 0 & \cdots & g^{(1)}(\lambda_i)
\end{bmatrix}$$

and hence by summing down the diagonals,

$$\|g(E_i)\|^2 = \sum_{s=0}^{k_j-1} ((k_j - s)/(s!)^2) \|g^{(s)}(\lambda_i)\|^2$$

and

$$\|g(A)\|^2 = c^2 \sum_{j=1}^{\alpha} \|g(E_j)\|^2 \leq c^2 \sum_{j=1}^{\alpha} \sum_{s=0}^{k_j-1} \frac{(k_j - s)}{(s!)^2} \|g^{(s)}(\lambda_j)\|^2,$$

and the proof is complete.

If $g(z) = z^p$ we may easily estimate $\|g(E_i)\|$. We distinguish two possibilities: (i) $p > k_j - 1$, (ii) $p \leq k_j - 1$. If (i), $\lambda_j = 0$ implies $\|g(E_j)\| = 0$, and $\lambda_j \neq 0$ implies

$$\|g(E_j)\|^2 = \lambda_j^{2p} \sum_{s=0}^{k_j-1} \left( \frac{p!}{s!} \right)^2 (k_j - s) \lambda_j^{-2s} \leq c_1 \lambda_j^{2p} p^{2k-2}.$$

If (ii), $\lambda_j = 0$ implies $\|g(E_j)\|^2 = (k_j - p)(p!)^2 \leq c_2 p^{2k-2}$, and again $\lambda_j \neq 0$ implies the same conclusion as in (i), where $c_1$ and $c_2$ depend only on the multiplicities and magnitudes of the $\lambda_j$. Thus

$$\|g(A)\|^2 \leq c_2 p^{2k-2} (c_1 + c_1 \sum_{j=1}^{n} |\lambda_j|^{2p}) \leq c_0 p^{2k-2} \sum_{j=1}^{n} |\lambda_j|^{2p}$$

since $A$ not nilpotent is equivalent to $\sum_{j=1}^{n} |\lambda_j|^{2p} \neq 0$. Noting that $\sum_{j=1}^{n} |g(\lambda_j)|^{2} = \sum_{j=1}^{n} |\lambda_j|^{2p}$, Gautschi’s result follows. We may employ the inequality to unify conveniently the familiar results concerning the asymptotic behavior of the system of ordinary differential equations $\dot{x} = Ax$, $A$ an $n$-square constant matrix, $x$ an $n$-column vector, with fundamental matrix of solutions $\exp (tA)$. Consider $g(z) = \exp (t z)$, $g^{(\nu)}(z) = t^\nu \exp (t z)$, $t \geq 0$, and the resulting inequality.
\[
\left( \sum_{j=1}^{n} | \exp (\lambda_j) |^2 \right)^{1/2} \leq \| \exp (tA) \| \\
\leq c \left( \sum_{j=1}^{n} \sum_{s=0}^{k_j-1} \frac{(k_j - s)}{(s!)^2} | t^s \exp (\lambda_j) |^2 \right)^{1/2}.
\]

Letting \( r_j = \text{Re} (\lambda_j) \) we have
\[
\left( \sum_{j=1}^{n} \exp (2tr_j) \right)^{1/2} \leq \| \exp (tA) \| \\
\leq c \left( \sum_{j=1}^{n} \sum_{s=0}^{k_j-1} \frac{(k_j - s)}{(s!)^2} t^{2s} \exp (2tr_j) \right)^{1/2}
\]
and we conclude immediately that:

(i) \( r_j > 0 \) for some \( j \) implies \( \lim_{t \to \infty} \| \exp (tA) \| = \infty \),
(ii) \( r_j < 0 \) for all \( j \) implies \( \lim_{t \to \infty} \| \exp (tA) \| = 0 \),
(iii) \( r_j \leq 0 \) for all \( j \) and \( r_j = 0 \) only if \( k_j = 1 \) implies \( \| \exp (tA) \| \)
is bounded on \([0, \infty)\).

In case \( A \) is nilpotent, \( \exp (tA) \) has as entries polynomials in \( t \).

REFERENCES


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