BI-REGULAR RINGS AND THE IDEAL LATTICE ISOMORPHISMS[1]

D. R. MORRISON

The purpose of this paper is to exhibit an isomorphism between a lattice of ideals (called the *ideals with local unit*) of an arbitrary ring R and the lattice of all ideals of a ring R^0, whose elements are the idempotent elements in the center of R. It is shown that every ideal of R is an ideal with local unit if and only if R is bi-regular in the sense of Arens and Kaplansky [1]. Thus in the bi-regular case, R and R^0 have isomorphic ideal lattices and all theorems on Boolean rings which can be stated in terms of the ideal lattice are extended immediately to bi-regular rings.

In particular, Arens and Kaplansky's extension to bi-regular rings of Stone's duality between Boolean rings and their structure spaces follows from the ideal *lattice isomorphisms*.

All numbers in brackets refer to the bibliography at the end of the paper.

1. **The idempotent Boolean ring, R^0, and its ideals.** The symbol R will be used throughout this paper to represent a *ring* and R^0 to represent the set of idempotent elements in the center of R. If $a, b \in R^0$, we define the symbol $\oplus$ by $a \oplus b = (a-b)^2$. The operation $\oplus$ will be called *idempotent addition*.

THEOREM 1.[2] *R^0 is a Boolean ring (called the idempotent Boolean ring of R) with respect to the operations of idempotent addition and multiplication. If e is a unit for R, then e is a unit for R^0.*

PROOF. The proof is merely the verification of the fact that the postulates for a Boolean ring are satisfied. The reader may supply the details.

LEMMA 1. *If I is an ideal of R, and $e \in I^0$, then e is in the center of R.*

PROOF. Since $e \in I^0$, $e^2 = e$, and e commutes with every element of I. Let $r \in R$. Then er, $re \in I$ and $er = e(er) = (er)e = e(re) = (re)e = re$. Thus e is in the center of R.

Presented to the Society, November 24, 1951; received by the editors November 6, 1953 and, in revised form, February 11, 1954.

[1] Done under ONR Contract no. N7-onr-434 Task Order III, Navy Department, Office of Naval Research.

[2] This theorem was proved by Foster [2] for R commutative, with unit.

LEMMA 2. *If I is an ideal of R, then $I^0 = I \cap R^0$.*

PROOF. This follows from the definition of I^0 and from Lemma 1.

DEFINITION. A ring R is said to be *with local unit* if R is the (set-theoretic) union of its ideals with unit; that is, every element of R is contained in an ideal of R possessing a unit.

Obviously, *every Boolean ring is a ring with local unit.*

THEOREM 2 (PRINCIPAL THEOREM). *If I is an ideal of R, then I^0 is an ideal of R^0. If I is an ideal with local unit, of R, then $I = RI^0$. If J is an ideal of R^0, then RJ is an ideal with local unit of R, and $(RJ)^0 = J$. Thus the set of ideals with local unit of R is a lattice, isomorphic to the lattice of all ideals of R^0, and in particular the set of ideals with unit of R is a sub-lattice, isomorphic to the lattice of principal ideals of R^0.*

PROOF. By Theorem 1, I^0 is a ring. By Lemma 2, $I^0 = I \cap R^0$. Both I and R^0 are closed under multiplication on either side by elements of R^0, so that I^0 is closed under such multiplication. Thus I^0 is an ideal of R^0.

Since I is an ideal of R, and $I^0 \subset I$, it follows that $RI^0 \subset I$. If I is a ring with local unit, then for any $x \in I$, there is an ideal of I, with unit e, containing x, and $e \in I^0$. Thus $x = xe \in RI^0$. Thus $I \subset RI^0$, $I = RI^0$.

Since J is in the center of R, $RJ = JR$. Clearly $RJ = JR$ is closed under multiplication on either side by elements of R. Also, for any $r_1j_1, r_2j_2 \in RJ$ ($r_i \in R$, $j_i \in J$), let $j = j_1 \oplus j_2 \oplus j_1j_2$. Then $j_1j = j_1$ and $j_2j = j_2$ and $j \in J$. Also $r_1j_1 - r_2j_2 = (r_1j_1 - r_2j_2)j \in RJ$. Thus RJ is closed under subtraction, and is therefore an ideal of R. For any $rj \in RJ$ ($r \in R$, $j \in J$), Rj is an ideal of RJ with unit, j, containing rj. Thus RJ is an ideal with local unit of R.

Clearly $J = J \cdot J \subset RJ$, hence $J \subset (RJ)^0$. Let $e \in (RJ)^0$. Then $e \in R^0$ and $e = rj$ ($r \in R$, $j \in J$). Thus $e = rj = rj^2 = ej \in R^0J \subset J$. Thus $(RJ)^0 \subset J$, $(RJ)^0 = J$.

It follows that the mappings $I \to I^0$ and $J \to RJ$ are reciprocal mappings between the set of ideals with local unit of R, and the set of all ideals of R^0. Since the mappings preserve inclusion, and are mutual inverses, they are $\subset$-isomorphisms. Thus the set of ideals with local unit of R is a lattice. By Theorem 1, the ideals with unit of R are mapped into ideals with unit of R^0, that is, into principal ideals of R^0. Conversely, if J has a unit j, then for any $rj_1 \in RJ$, $r \in R$, $j_1 \in J$, $(rj_1) \cdot j = rj_1$, and j is a unit for RJ. Thus RJ is an ideal with unit, of R. This proves the theorem.

2. **Bi-regular rings.** The lattice isomorphisms of Theorem 2 will be particularly useful in the class of rings R, whose ideals are *all* with local unit. For in this case, we can study the ideal lattice of R by studying the ideal lattice of the relatively simpler ring R^0. The purpose of the next theorem is to identify this class of rings with the class which Arens and Kaplansky [1] have called *bi-regular*.

THEOREM 3. *The following conditions upon a ring R are equivalent.*

(1) *Every ideal of R is with local unit.*

(2) *Every principal ideal of R has a unit.*

(3)[3] *Every principal ideal of R is generated by an idempotent in the center of R.*

PROOF. Clearly (1) implies (2). If (2) holds, and if $(r)_R$ is a principal ideal of R, then $(r)_R$ has a unit r^0. By Lemma 1, r^0 is in the center of R. Clearly $(r)_R=(r^0)_R$ and r^0 is idempotent and in the center of R. Thus (2) implies (3). Finally, if (3) holds, then every principal ideal $(r)_R$ has the form $(r^0)_R$, where r^0 is idempotent and in the center of R. It follows that r^0 is a unit for $(r)_R$, and since any ideal I of R is the (set-theoretic) union of the principal ideals it contains, I is with local unit. Thus (3) implies (1).

3. **The center of a bi-regular ring.** In this section we shall use the symbol R^c to represent the *center* of a ring R. We shall show that the ideal lattice isomorphism between R and R^0 extends also to R^c. This extends the classical theorems "The center of a ring *with minimum condition*, and without radical, is without radical," and "The center of a direct sum of two-sided ideals is the direct sum of their centers." See [5, §118].

THEOREM 4. *If R is a bi-regular ring, then R^c is bi-regular.*

PROOF. Let $r\in R^c$. Then the principal ideal $(r)_R$ of R is equal to Rr, and has a unit $r^0=sr$ for some $s\in R$. Let $t=sr^0$. Then $tr=sr^0r=r^0$ and $tr^0=s(r^0)^2=sr^0=t$. Thus for any $x\in R$, $tx=tr^0x=txr^0=txrt=trxt=r^0xt=xtr^0=xt$. Thus $t\in R^c$. It follows that $r^0=tr\in R^cr$. But R^cr is the principal ideal of R^c, containing r, and has a unit r^0. Thus R^c is bi-regular.

LEMMA 3. *If I is an ideal of a bi-regular ring R, then $I^c=I\cap R^c$ and thus I^c is an ideal of R^c.*

[3] This is Arens and Kaplansky's definition of a bi-regular ring. They remark further that in the presence of commutativity or descending chain condition, bi-regularity is equivalent to regularity. Bi-regularity and regularity are independent generalizations of classical semisimplicity.

PROOF. Obviously $I^c \supseteq (I \cap R^c)$. If $x \in I^c$ then for any $r \in R$, $rx = rx^0x = xrx^0 = xx^0r = xr$, where x^0 is the unit of $(x)_R$. Hence $I^c \subseteq (I \cap R^c)$.

THEOREM 5. *If R is bi-regular, then the mappings $I \to I^c$ and $J \to RJ$ are reciprocal lattice isomorphisms between the ideal lattices of R and R^c, and they map principal ideals into principal ideals.*

PROOF. Obviously $(R^c)^0 = R^0$. By applying Theorem 2 to R and to R^c, we obtain isomorphisms between the ideal lattices of R and of R^0, and of R^c and R^0. Composing these isomorphisms, we have the required isomorphisms. For any ideal I of R, and any ideal J of R^c, $I \to I^0 \to R^cI^0 = R^c(I^c)^0 = I^c$ and $J \to J^0 \to RJ^0 = (RR^c)J^0 = R(R^cJ^0) = RJ$. Since the mappings are compositions of reciprocal lattice isomorphisms which carry principal ideals into principal ideals, they have the required properties.

Using the well known duality between homomorphisms and ideals, a correspondence can be established between homomorphisms of a bi-regular ring R, and homomorphisms of R^0 and of R^c. This in turn induces a correspondence between subdirect representations (see [3]) and structure spaces (see [4]) of the three rings. With these tools, all theorems on Boolean rings or commutative bi-regular rings, which can be stated in terms of ideals, principal ideals, homomorphisms, subdirect representations, or structure spaces, are extended immediately to bi-regular rings.

BIBLIOGRAPHY

1. R. Arens and I. Kaplansky, *Topological representations of algebras*, Trans. Amer. Math. Soc. vol. 63 (1949) pp. 457–481.

2. A. S. Foster, *The idempotent elements of a commutative ring*, Duke Math. J. vol. 12 (1945) pp. 143–152.

3. N. H. McCoy, *Subdirect sums of rings*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 856–877.

4. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481.

5. B. L. van der Waerden, *Moderne Algebra*, vol. 2.

TULANE UNIVERSITY