BI-REGULAR RINGS AND THE IDEAL LATTICE ISOMORPHISMS

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The purpose of this paper is to exhibit an isomorphism between a lattice of ideals (called the ideals with local unit) of an arbitrary ring $R$ and the lattice of all ideals of a ring $R^0$, whose elements are the idempotent elements in the center of $R$. It is shown that every ideal of $R$ is an ideal with local unit if and only if $R$ is bi-regular in the sense of Arens and Kaplansky [1]. Thus in the bi-regular case, $R$ and $R^0$ have isomorphic ideal lattices and all theorems on Boolean rings which can be stated in terms of the ideal lattice are extended immediately to bi-regular rings.

In particular, Arens and Kaplansky's extension to bi-regular rings of Stone's duality between Boolean rings and their structure spaces follows from the ideal lattice isomorphisms.

All numbers in brackets refer to the bibliography at the end of the paper.

1. The idempotent Boolean ring, $R^0$, and its ideals. The symbol $R$ will be used throughout this paper to represent a ring and $R^0$ to represent the set of idempotent elements in the center of $R$. If $a, b \in R^0$, we define the symbol $\oplus$ by $a \oplus b = (a - b)^2$. The operation $\oplus$ will be called idempotent addition.

Theorem 1. $R^0$ is a Boolean ring (called the idempotent Boolean ring of $R$) with respect to the operations of idempotent addition and multiplication. If $e$ is a unit for $R$, then $e$ is a unit for $R^0$.

Proof. The proof is merely the verification of the fact that the postulates for a Boolean ring are satisfied. The reader may supply the details.

Lemma 1. If $I$ is an ideal of $R$, and $e \in I^0$, then $e$ is in the center of $R$.

Proof. Since $e \in I^0$, $e^2 = e$, and $e$ commutes with every element of $I$. Let $r \in R$. Then $er, re \in I$ and $er = e(er) = (er)e = e(re) = (re)e = re$. Thus $e$ is in the center of $R$.

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This theorem was proved by Foster [2] for $R$ commutative, with unit.
Lemma 2. If $I$ is an ideal of $R$, then $I^0 = I \cap R^0$.

Proof. This follows from the definition of $I^0$ and from Lemma 1.

Definition. A ring $R$ is said to be with local unit if $R$ is the (set-theoretic) union of its ideals with unit; that is, every element of $R$ is contained in an ideal of $R$ possessing a unit.

Obviously, every Boolean ring is a ring with local unit.

Theorem 2 (Principal Theorem). If $I$ is an ideal of $R$, then $I^0$ is an ideal of $R^0$. If $I$ is an ideal with local unit, of $R$, then $I = RI^0$. If $J$ is an ideal of $R^0$, then $RJ$ is an ideal with local unit of $R$, and $(RJ)^0 = J$. Thus the set of ideals with local unit of $R$ is a lattice, isomorphic to the lattice of all ideals of $R^0$, and in particular the set of ideals with unit of $R$ is a sub-lattice, isomorphic to the lattice of principal ideals of $R^0$.

Proof. By Theorem 1, $I^0$ is a ring. By Lemma 2, $I^0 = I \cap R^0$. Both $I$ and $R^0$ are closed under multiplication on either side by elements of $R^0$, so that $I^0$ is closed under such multiplication. Thus $I^0$ is an ideal of $R^0$.

Since $I$ is an ideal of $R$, and $I^0 \subseteq I$, it follows that $RI^0 \subseteq I$. If $I$ is a ring with local unit, then for any $x \in I$, there is an ideal of $I$, with unit $e$, containing $x$, and $e \in I^0$. Thus $x = xe \in RI^0$. Thus $I \subseteq RI^0$, $I = RI^0$.

Since $J$ is in the center of $R$, $RJ = JR$. Clearly $RJ = JR$ is closed under multiplication on either side by elements of $R$. Also, for any $r_1j_1, r_2j_2 \in RJ$ ($r_1 \in R$, $j_1 \in J$), let $j = j_1 \oplus j_2 \oplus j_1j_2$. Then $j_1j = j_1$ and $j_2j = j_2$ and $j \in J$. Also $r_1j_1 - r_2j_2 = (r_1j_1 - r_2j_2)j \in RJ$. Thus $RJ$ is closed under subtraction, and is therefore an ideal of $R$. For any $rj \in RJ$ ($r \in R$, $j \in J$), $Rj$ is an ideal of $RJ$ with unit, $j$, containing $rj$. Thus $RJ$ is an ideal with local unit of $R$.

Clearly $J = J \cdot J \subseteq RJ$, hence $J \subseteq (RJ)^0$. Let $e \in (RJ)^0$. Then $e \in R^0$ and $e = rj$ ($r \in R$, $j \in J$). Thus $e = rj = ej \in R^0J \subseteq J$. Thus $(RJ)^0 \subseteq J$, $(RJ)^0 = J$.

It follows that the mappings $I \rightarrow I^0$ and $J \rightarrow RJ$ are reciprocal mappings between the set of ideals with local unit of $R$, and the set of all ideals of $R^0$. Since the mappings preserve inclusion, and are mutual inverses, they are $\subseteq$-isomorphisms. Thus the set of ideals with local unit of $R$ is a lattice. By Theorem 1, the ideals with unit of $R$ are mapped into ideals with unit of $R^0$, that is, into principal ideals of $R^0$. Conversely, if $J$ has a unit $j$, then for any $rj_1 \in RJ$, $r \in R$, $j_1 \in J$, $(rj_1)j = rj_1$, and $j$ is a unit for $RJ$. Thus $RJ$ is an ideal with unit, of $R$. This proves the theorem.
2. **Bi-regular rings.** The lattice isomorphisms of Theorem 2 will be particularly useful in the class of rings \( R \), whose ideals are all with local unit. For in this case, we can study the ideal lattice of \( R \) by studying the ideal lattice of the relatively simpler ring \( R^0 \). The purpose of the next theorem is to identify this class of rings with the class which Arens and Kaplansky [1] have called \textit{bi-regular}.

**Theorem 3.** The following conditions upon a ring \( R \) are equivalent.

1. Every ideal of \( R \) is with local unit.
2. Every principal ideal of \( R \) has a unit.
3. Every principal ideal of \( R \) is generated by an idempotent in the center of \( R \).

**Proof.** Clearly (1) implies (2). If (2) holds, and if \((r)R\) is a principal ideal of \( R \), then \((r)R\) has a unit \( r^0 \). By Lemma 1, \( r^0 \) is in the center of \( R \). Clearly \((r)R = (r^0)R\) and \( r^0 \) is idempotent and in the center of \( R \). Thus (2) implies (3). Finally, if (3) holds, then every principal ideal \((r)R\) has the form \((r^0)R\), where \( r^0 \) is idempotent and in the center of \( R \). It follows that \( r^0 \) is a unit for \((r)R\), and since any ideal \( I \) of \( R \) is the (set-theoretic) union of the principal ideals it contains, \( I \) is with local unit. Thus (3) implies (1).

3. **The center of a bi-regular ring.** In this section we shall use the symbol \( R^e \) to represent the center of a ring \( R \). We shall show that the ideal lattice isomorphism between \( R \) and \( R^0 \) extends also to \( R^e \). This extends the classical theorems "The center of a ring with minimum condition, and without radical, is without radical," and "The center of a direct sum of two-sided ideals is the direct sum of their centers." See [5, §118].

**Theorem 4.** If \( R \) is a bi-regular ring, then \( R^e \) is bi-regular.

**Proof.** Let \( r \in R^e \). Then the principal ideal \((r)R\) of \( R \) is equal to \( Rr \), and has a unit \( r^0 = sr \) for some \( s \in R \). Let \( t = sr^0 \). Then \( tr = sr^0r = r^0 \) and \( tr^0 = s(r^0)^2 = sr^0 = t \). Thus for any \( x \in R \), \( tx = tr^0x = txr^0 = txr = txt = trxt = r^0xt = xtr^0 = xt \). Thus \( t \in R^e \). It follows that \( r^0 = tr \in R^e \). But \( R^e r \) is the principal ideal of \( R^e \), containing \( r \), and has a unit \( r^0 \). Thus \( R^e \) is bi-regular.

**Lemma 3.** If \( I \) is an ideal of a bi-regular ring \( R \), then \( I^e = I \cap R^e \) and thus \( I^e \) is an ideal of \( R^e \).

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*This is Arens and Kaplansky's definition of a bi-regular ring. They remark further that in the presence of commutativity or descending chain condition, bi-regularity is equivalent to regularity. Bi-regularity and regularity are independent generalizations of classical semisimplicity.*
Proof. Obviously $I^o \supseteq (I \cap R^o)$. If $x \in I^o$ then for any $r \in R$, $rx = rx^o x = xrx^o = xx^o r = xr$, where $x^o$ is the unit of $(x)_R$. Hence $I^o \subseteq (I \cap R^o)$.

**Theorem 5.** If $R$ is bi-regular, then the mappings $I \rightarrow I^c$ and $J \rightarrow R^c J$ are reciprocal lattice isomorphisms between the ideal lattices of $R$ and $R^c$, and they map principal ideals into principal ideals.

Proof. Obviously $(R^c)^o = R^o$. By applying Theorem 2 to $R$ and to $R^c$, we obtain isomorphisms between the ideal lattices of $R$ and of $R^o$, and of $R^c$ and $R^o$. Composing these isomorphisms, we have the required isomorphisms. For any ideal $I$ of $R$, and any ideal $J$ of $R^c$, $I \rightarrow I^o \rightarrow R^c I^o = R^c (I^c)^o = I^c$ and $J \rightarrow J^o \rightarrow R J^o = (R R^c) J^o = R (R^c J^o) = R J$. Since the mappings are compositions of reciprocal lattice isomorphisms which carry principal ideals into principal ideals, they have the required properties.

Using the well known duality between homomorphisms and ideals, a correspondence can be established between homomorphisms of a bi-regular ring $R$, and homomorphisms of $R^0$ and of $R^c$. This in turn induces a correspondence between subdirect representations (see [3]) and structure spaces (see [4]) of the three rings. With these tools, all theorems on Boolean rings or commutative bi-regular rings, which can be stated in terms of ideals, principal ideals, homomorphisms, subdirect representations, or structure spaces, are extended immediately to bi-regular rings.

**Bibliography**