ON THE INTERSECTIONS OF THE COMPONENTS OF A DIFFERENCE POLYNOMIAL

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The purpose of this note is to prove the following theorem:

Solutions common to two distinct components\(^1\) of the manifold of a difference polynomial annul the separants of the polynomial.

We begin by considering a field \(K\), not necessarily a difference field, and a set of polynomials \(F_1, F_2, \ldots, F_p\) in \(K[u_1, \ldots, u_q; x_1, \ldots, x_p]\), the \(u_i\) and \(x_j\) being indeterminates, where for each \(j, j = 1, \ldots, p-1\), \(F_j\) is free of the \(x_k, k > j\). We shall show that any zero of \(F_1, \ldots, F_p\) which annuls no formal partial derivative \(\partial F_j/\partial x_j\) belongs to just one component of \(\{F_1, \ldots, F_p\}_0\).\(^2\) Furthermore, this component is of dimension \(q\).

Proof. Let \(u_i = y_i, i = 1, \ldots, q; x_j = \alpha_j, j = 1, \ldots, p\), be a zero of \(F_1, \ldots, F_p\) which annuls no \(\partial F_j/\partial x_j\). If \(y_i, \ldots, y_q; \alpha_i, \ldots, \alpha_p\) is a zero of \(F_1, \ldots, F_p\) which specializes to \(y_i, \ldots, y_q; \alpha_i, \ldots, \alpha_p\), then this zero too annuls no \(\partial F_j/\partial x_j\). It follows from this that \(\alpha_i\) is algebraic over \(K(\gamma'_1, \ldots, \gamma'_q)\), and that for each \(k, 1 < k \leq p\), \(\alpha'_k\) is algebraic over \(K(\gamma'_1, \ldots, \gamma'_q; \alpha'_1, \ldots, \alpha'_{k-1})\). This implies that a component of the manifold of \(\{F_1, \ldots, F_p\}_0\) containing \(y_1, \ldots, y_q; \alpha_1, \ldots, \alpha_p\) is of dimension at most \(q\).

We let \(u_i = t_i + \gamma_i, i = 1, \ldots, q; x_j = \alpha_j + h_j, j = 1, \ldots, p\). Here the \(t_i\) denote new indeterminates and the \(h_j\) certain formal series in positive integral powers of the \(t_i\). We shall show that the \(h_j\) may be so chosen that these substitutions annul \(F_1, \ldots, F_p\). In fact, the lemma proved in [3] shows that for each \(k, 1 \leq k \leq p\), we may annul \(F_k\) by substitutions \(u_i = t_i + \gamma_i, i = 1, \ldots, p, x_j = s_j + \alpha_j, j < k, x_k = \alpha_k + h'_k\), where the \(s_j, j = 1, \ldots, p\), are new indeterminates, and \(h'_k\) is a formal series in positive integral powers of the \(t_i\) and \(s_j, j < k\). For \(h_1\) we take \(h'_1\); for \(h_2\) we take the result of replacing \(s_1\) in \(h'_2\) by \(h'_1\), and so on.

With the \(h_j\) as described let \(\Sigma\) denote the set of polynomials in \(K[u_1, \ldots, u_q; x_1, \ldots, x_p]\) which are annulled by the above substitutions. Evidently \(\Sigma\) is a prime p. i. (polynomial ideal). Its dimen-

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1 The term “component,” not previously defined for difference manifolds, is to have the expected meaning: a component is a maximal irreducible submanifold of a manifold. For definitions of other terms and symbols see [2; 3; 4].

2 As in Chapter IV of [1] this notation indicates the perfect polynomial ideal generated by \(F_1, F_2, \ldots, F_p\).
sion is $q$ and the $u_i$ form a parametric set. For evidently $\Sigma$ can contain no polynomial in the $u_i$ alone, while the conclusion of the preceding paragraph but one shows that its dimension cannot exceed $q$. The result of that paragraph also shows that no component of $\{ F_1, \ldots, F_p \}_0$ can properly contain the manifold of $\Sigma$, for then its dimension would exceed $q$. Hence this manifold is itself a component of $\{ F_1, \ldots, F_p \}_0$.

Let $\mathcal{M}$ be a component of $\{ F_1, \ldots, F_p \}_0$ which contains $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$, and let $\Lambda$ be the prime p. i. in $K[u_1, \ldots, u_q; x_1, \ldots, x_p]$ whose manifold is $\mathcal{M}$. We must show that $\Lambda$ is $\Sigma$. If $\Lambda$ is of dimension 0 then, because $\Sigma$ vanishes for a zero of $\Lambda$, and every zero must be a generic zero, $\Sigma$ is contained in $\Lambda$. Since the manifolds of both are components of the same manifold, it follows that $\Lambda = \Sigma$ (and that $q = 0$). We suppose that $\Lambda$ is of positive dimension, and that $\Lambda$ and $\Sigma$ are distinct. Then, since $\Lambda$ cannot contain $\Sigma$, there is a polynomial $P$ in $\Sigma$ which is not in $\Lambda$. Then $\Lambda$ possesses a zero not annulling $P$ of the form

\begin{equation}
\begin{aligned}
u_i &= \gamma_i + g_i, \quad i = 1, \ldots, q; \\
x_j &= \alpha_j + f_j, \quad j = 1, \ldots, p,
\end{aligned}
\end{equation}

where the $g_i$ and the $f_j$ are series in positive integral powers of a parameter $t$.

It is evident that (1) is a zero of $F_1, \ldots, F_p$. We may also obtain a zero of these polynomials of the form

\begin{equation}
\begin{aligned}
u_i &= \gamma_i + g_i, \quad i = 1, \ldots, q; \\
x_j &= \alpha_j + f_j', \quad j = 1, \ldots, p,
\end{aligned}
\end{equation}

where the $f_j'$ are again series in positive integral powers of $t$, and each $f_j'$ is obtained by replacing the $t_i, i = 1, \ldots, p$, in $h_j$ by the corresponding $g_i$. It is evident from the manner of formation of (2) that it is a zero of $\Sigma$.

We replace the $u_i$ in $F_1$ by $\gamma_i + g_i, i = 1, \ldots, q$. There results a polynomial $\bar{F}_1$ in $x_1$ with coefficients power series in $t$. $\bar{F}_1$ vanishes, but its formal derivative $d\bar{F}_1/dx_1$ does not, when we put $t = 0$, $x_1 = \alpha_1$. It follows that there is a unique series $f''_1$ in positive integral powers of $t$ such that $x_1 = \alpha_1 + f''_1$ is a solution of $\bar{F}_1 = 0$. We now replace the $u_i, i = 1, \ldots, q$, and $x_1$ in $F_2$ by $\gamma_i + g_i$ and $\alpha_1 + f''_1$ respectively to obtain a polynomial $\bar{F}_2$ in $x_2$ with coefficients power series in $t$. As before, we see that $\bar{F}_2 = 0$ possesses a solution $x_2 = \alpha_2 + f''_2$, where $f''_2$ is a series in positive integral powers of $t$. This series is unique. Continuing in this way we find uniquely determined $f''_j, j = 1, \ldots, p$, which are series in positive integral powers of $t$ such that $u_i = \gamma_i + g_i$,
\[ i = 1, \ldots, q; x_j = \alpha_j + \beta_j, \ j = 1, \ldots, p, \text{ is a zero of } F_1, \ldots, F_p. \]

The uniqueness of the \( \beta_j \) shows that (1) and (2) are identical. Hence (1) annuls \( \Sigma \), and, in particular, it annuls \( P \). We have thus obtained a contradiction. This completes the proof of our statement concerning the zeros of \( F_1, \ldots, F_p \).

Now let \( \mathcal{J} \) be a difference field and \( A \) a polynomial of \( \mathcal{J}\{y_1, \ldots, y_n\} \). We shall prove the theorem stated at the beginning of this note. We may suppose that a transform of some \( y_i \), say of \( y_n \), appears effectively in \( A \). Let \( y_i = \alpha_i, \ i = 1, \ldots, n, \) be a zero of \( A \). It will suffice to assume that the \( \alpha_i \) are not a zero of the \( y_n \)-separant of \( A \) and show that this implies that only one component of the manifold of \( A \) contains the \( \alpha_i \).

It is evident that the \( \alpha_i \) must annul just one irreducible factor, say \( F \), of \( A \), and do not annul the \( y_n \)-separant of \( F \). Hence we need merely show that the \( \alpha_i \) are contained in only one component of the manifold of \( F \). We shall suppose that this is not so and obtain a contradiction. We assume first that \( F \) is of equal order and effective order in \( y_n \).

Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) denote two distinct components of the manifold of \( F \), each containing the \( \alpha_i \). Let \( \Sigma_1 \) and \( \Sigma_2 \) denote the corresponding reflexive prime difference ideals. We denote by \( h \) the order of \( F \) in \( y_n \). Since the \( \alpha_i \) do not annul the \( y_n \)-separant of \( F \), \( y_i, \ldots, y_{n-1} \) constitute a parametric set for both \( \Sigma_1 \) and \( \Sigma_2 \), and these ideals are both of order \( h \) in \( y_n \).

We choose an integer \( m \) such that the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) do not constitute the beginning of a characteristic sequence of \( \Sigma_2 \). Let \( \Sigma_1^m \) and \( \Sigma_2^m \) denote the sets consisting of those polynomials of \( \Sigma_1 \) and \( \Sigma_2 \) respectively which involve the \( y_n^k, 0 \leq k \leq m+h \), and a finite subset \( S \) of the \( y_{ij}, i < n \). \( S \) is to include all those \( y_{ij}, i < n \), which appear effectively, or whose transforms appear effectively, in \( F, F_1, \ldots, F_m \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_2 \).

\( \Sigma_1^m \) and \( \Sigma_2^m \) may be regarded as prime \( \mathcal{P} \) i.'s in the ring \( \mathcal{J}[S, y_n^0, y_n^1, \ldots, y_n^{m+h}] \). The \( y_{ij} \) of \( S \) and the \( y_n^k, k < h \), constitute a parametric set for both \( \Sigma_1^m \) and \( \Sigma_2^m \). Let \( s \) denote the number of indeterminates in this parametric set.

Our earlier result concerning polynomial ideals shows that there is a unique component \( \mathcal{M} \) of the manifold of \( \{F, F_1, \ldots, F_m\} \), regarded as an ideal of \( \mathcal{J}[S, y_n^0, y_n^1, \ldots, y_n^{m+h}] \), which contains the zero \( y_{ij} = \alpha_{ij} \) of this ideal. The dimension of \( \mathcal{M} \) is \( s \), for \( s \) corresponds to \( q \) of the earlier proof.

Now both \( \Sigma_1^m \) and \( \Sigma_2^m \) contain \( \{F, F_1, \ldots, F_m\} \), while both have
the zero \( y_{ij} = \alpha_{ij} \). Hence their manifolds are in \( \mathcal{M} \). Since their manifolds are of dimension \( s \), however, they must coincide with \( \mathcal{M} \). Hence \( \Sigma_{1m} \) and \( \Sigma_{2m} \) are identical. But \( m \) was chosen so that \( \Sigma_{1m} \) contains a polynomial which is not in \( \Sigma_{2m} \), namely one of the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \). We have obtained a contradiction. This completes the proof of the theorem in the case that \( F \) is of equal order and effective order in \( y_n \).

If the order of \( F \) in \( y_n \) exceeds its effective order by \( d > 0 \), we replace each \( y_{nk} \) in \( F \) by \( z_{k-d} \), where \( z \) is a new indeterminate, and subscripts attached to \( z \) denote transforming. \( F \) goes into an irreducible polynomial \( \overline{F} \) which is of equal order and effective order in \( z \).

Evidently each component \( \overline{\mathcal{M}} \) of the manifold of \( \overline{F} \) corresponds to a unique component \( \mathcal{M} \) of the manifold of \( F \), and, conversely, each component of the manifold of \( F \) is obtained from a unique component of the manifold of \( \overline{F} \). The correspondence may be described as follows: each solution in \( \overline{\mathcal{M}} \) is obtained from a solution in \( \overline{\mathcal{M}} \) by leaving unchanged the elements assigned as values to \( y_1, \ldots, y_{n-1} \), and assigning to \( y_n \) an element whose \( d \)th transform is the element assigned as the value of \( z \) in \( \overline{\mathcal{M}} \). This correspondence carries solutions common to two components of the manifold of \( F \) into solutions common to two components of the manifold of \( \overline{F} \). Solutions annulling the \( y_n \)-separant of \( F \) correspond to solutions annulling the \( z \)-separant of \( \overline{F} \).

The preceding proof shows that the theorem stated at the beginning of this note holds for \( \overline{F} \). The correspondence just described shows that its truth for \( \overline{F} \) implies its truth for \( F \). Hence it is true in general.

References


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