ON THE INTERSECTIONS OF THE COMPONENTS OF A DIFFERENCE POLYNOMIAL

RICHARD M. COHN

The purpose of this note is to prove the following theorem:

Solutions common to two distinct components of the manifold of a difference polynomial annul the separants of the polynomial.

We begin by considering a field \( K \), not necessarily a difference field, and a set of polynomials \( F_1, F_2, \ldots, F_p \) in \( K[u_1, \ldots, u_q; x_1, \ldots, x_p] \), the \( u_i \) and \( x_j \) being indeterminates, where for each \( j, j = 1, \ldots, p - 1, \) \( F_j \) is free of the \( x_k, k > j \). We shall show that any zero of \( F_1, \ldots, F_p \) which annuls no formal partial derivative \( \partial F_j/\partial x_j \) belongs to just one component of \( \{ F_1, \ldots, F_p \} \). Furthermore, this component is of dimension \( q \).

Proof. Let \( \gamma_1, \gamma_2, \ldots, \gamma_q, \alpha_1, \ldots, \alpha_p \) be a zero of \( F_1, \ldots, F_p \) which annuls no \( \partial F_j/\partial x_j \). If \( \gamma_1', \gamma_2', \ldots, \gamma_q', \alpha_1', \ldots, \alpha_p' \) is a zero of \( F_1, \ldots, F_p \) which specializes to \( \gamma_1, \gamma_2, \ldots, \gamma_q, \alpha_1, \ldots, \alpha_p \), then this zero too annuls no \( \partial F_j/\partial x_j \). It follows from this that \( \alpha_i' \) is algebraic over \( K(\gamma_1', \gamma_2', \ldots, \gamma_q') \), and that for each \( k, 1 < k \leq p \), \( \alpha_k' \) is algebraic over \( K(\gamma_1', \gamma_2', \ldots, \alpha_k', \ldots, \alpha_{k-1}') \). This implies that a component of the manifold of \( \{ F_1, \ldots, F_p \} \) containing \( \gamma_1, \gamma_2, \ldots, \gamma_q, \alpha_1, \ldots, \alpha_p \) is of dimension at most \( q \).

We let \( u_i = u_i + \gamma_i, i = 1, \ldots, q; x_j = \alpha_j + h_j, j = 1, \ldots, p \). Here the \( t_i \) denote new indeterminates and the \( h_i \) certain formal series in positive integral powers of the \( t_i \). We shall show that these substitutions annul \( F_1, \ldots, F_p \). In fact, the lemma proved in [3] shows that for each \( k, 1 \leq k \leq p \), we may annul \( F_k \) by substitutions \( u_i = t_i + \gamma_i, i = 1, \ldots, p, x_j = s_j + \alpha_j, j < k, x_k = \alpha_k + h_k' \), where the \( s_j, j = 1, \ldots, p \), are new indeterminates, and \( h_k' \) is a formal series in positive integral powers of the \( t_i \) and \( s_j, j < k \). For \( h_1 \) we take \( h_1' \); for \( h_2 \) we take the result of replacing \( s_1 \) in \( h_2' \) by \( h_1' \), and so on.

With the \( h_j \) as described let \( \Sigma \) denote the set of polynomials in \( K[u_1, \ldots, u_q; x_1, \ldots, x_p] \) which are annulled by the above substitutions. Evidently \( \Sigma \) is a prime p. i. (polynomial ideal). Its dimen-

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1 The term “component,” not previously defined for difference manifolds, is to have the expected meaning: a component is a maximal irreducible submanifold of a manifold. For definitions of other terms and symbols see [2; 3; 4].

2 As in Chapter IV of [1] this notation indicates the perfect polynomial ideal generated by \( F_1, F_2, \ldots, F_p \).
sion is $q$ and the $u_i$ form a parametric set. For evidently $\Sigma$ can contain no polynomial in the $u_i$ alone, while the conclusion of the preceding paragraph but one shows that its dimension cannot exceed $q$. The result of that paragraph also shows that no component of $\{ F_1, \ldots, F_p \}_0$ can properly contain the manifold of $\Sigma$, for then its dimension would exceed $q$. Hence this manifold is itself a component of $\{ F_1, \ldots, F_p \}_0$.

Let $\mathcal{M}$ be a component of $\{ F_1, \ldots, F_p \}_0$ which contains $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$, and let $\Lambda$ be the prime p. i. in $K[u_1, \ldots, u_q; x_1, \ldots, x_p]$ whose manifold is $\mathcal{M}$. We must show that $\Lambda$ is $\Sigma$. If $\Lambda$ is of dimension 0 then, because $\Sigma$ vanishes for a zero of $\Lambda$, and every zero must be a generic zero, $\Sigma$ is contained in $\Lambda$. Since the manifolds of both are components of the same manifold, it follows that $\Lambda = \Sigma$ (and that $q = 0$). We suppose that $\Lambda$ is of positive dimension, and that $\Lambda$ and $\Sigma$ are distinct. Then, since $\Lambda$ cannot contain $\Sigma$, there is a polynomial $P$ in $\Sigma$ which is not in $\Lambda$. Then $\Lambda$ possesses a zero not annulling $P$ of the form

\begin{align*}
&u_i = \gamma_i + g_i, \quad i = 1, \ldots, q; \\
x_j = \alpha_j + f_j, \quad j = 1, \ldots, p,
\end{align*}

where the $g_i$ and the $f_j$ are series in positive integral powers of a parameter $t$.

It is evident that (1) is a zero of $F_1, \ldots, F_p$. We may also obtain a zero of these polynomials of the form

\begin{align*}
&u_i = \gamma_i + g_i, \quad i = 1, \ldots, q; \\
x_j = \alpha_j + f_j', \quad j = 1, \ldots, p,
\end{align*}

where the $f_j'$ are again series in positive integral powers of $t$, and each $f_j'$ is obtained by replacing the $t_i, i = 1, \ldots, p$, in $h_j$ by the corresponding $g_i$. It is evident from the manner of formation of (2) that it is a zero of $\Sigma$.

We replace the $u_i$ in $F_1$ by $\gamma_i + g_i, i = 1, \ldots, q$. There results a polynomial $\bar{F}_1$ in $x_1$ with coefficients power series in $t$. $\bar{F}_1$ vanishes, but its formal derivative $d\bar{F}_1/dx_1$ does not, when we put $t = 0$, $x_1 = \alpha_1$. It follows that there is a unique series $f_1''$ in positive integral powers of $t$ such that $x_1 = \alpha_1 + f_1''$ is a solution of $\bar{F}_1 = 0$. We now replace the $u_i, i = 1, \ldots, q$, and $x_1$ in $F_2$ by $\gamma_i + g_i$ and $\alpha_1 + f_1''$ respectively to obtain a polynomial $\bar{F}_2$ in $x_2$ with coefficients power series in $t$. As before, we see that $\bar{F}_2 = 0$ possesses a solution $x_2 = \alpha_2 + f_2''$, where $f_2''$ is a series in positive integral powers of $t$. This series is unique. Continuing in this way we find uniquely determined $f_j'', j = 1, \ldots, p$, which are series in positive integral powers of $t$ such that $u_i = \gamma_i + g_i$. 

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$i = 1, \ldots, q; x_j = \alpha_j + f''_j, j = 1, \ldots, p$, is a zero of $F_1, \ldots, F_p$.

The uniqueness of the $f''_j$ shows that (1) and (2) are identical. Hence (1) annihilates $\Sigma$, and, in particular, it annihilates $P$. We have thus obtained a contradiction. This completes the proof of our statement concerning the zeros of $F_1, \ldots, F_p$.

Now let $\mathcal{F}$ be a difference field and $A$ a polynomial of $\mathcal{F}\{y_1, \ldots, y_n\}$. We shall prove the theorem stated at the beginning of this note. We may suppose that a transform of some $y_i$, say of $y_n$, appears effectively in $A$. Let $y_i = \alpha_i, i = 1, \ldots, n$, be a zero of $A$. It will suffice to assume that the $\alpha_i$ are not a zero of the $y_n$-separant of $A$ and show that this implies that only one component of the manifold of $A$ contains the $\alpha_i$.

It is evident that the $\alpha_i$ must annihilate just one irreducible factor, say $F$, of $A$, and do not annihilate the $y_n$-separant of $F$. Hence we need merely show that the $\alpha_i$ are contained in only one component of the manifold of $F$. We shall suppose that this is not so and obtain a contradiction. We assume first that $F$ is of equal order and effective order in $y_n$.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote two distinct components of the manifold of $F$, each containing the $\alpha_i$. Let $\Sigma_1$ and $\Sigma_2$ denote the corresponding reflexive prime difference ideals. We denote by $h$ the order of $F$ in $y_n$. Since the $\alpha_i$ do not annihilate the $y_n$-separant of $F$, $y_1, \ldots, y_{n-1}$ constitute a parametric set for both $\Sigma_1$ and $\Sigma_2$, and these ideals are both of order $h$ in $y_n$.

We choose an integer $m$ such that the first $m+1$ polynomials of a characteristic sequence of $\Sigma_1$ do not constitute the beginning of a characteristic sequence of $\Sigma_2$. Let $\Sigma_{1m}$ and $\Sigma_{2m}$ denote the sets consisting of those polynomials of $\Sigma_1$ and $\Sigma_2$ respectively which involve the $y_k^n$, $0 \leq k \leq m+h$, and a finite subset $S$ of the $y_{ij}, i < n$. $S$ is to include all those $y_{ij}, i < n$, which appear effectively, or whose transforms appear effectively, in $F, F_1, \ldots, F_m$ or in the first $m+1$ polynomials of a characteristic sequence of $\Sigma_1$ or in the first $m+1$ polynomials of a characteristic sequence of $\Sigma_2$.

$\Sigma_{1m}$ and $\Sigma_{2m}$ may be regarded as primes in the ring $\mathcal{F}[S, y_n^0, y_n^1, \ldots, y_n^{m+h}]$. The $y_{ij}$ of $S$ and the $y_{nk}, k < h$, constitute a parametric set for both $\Sigma_{1m}$ and $\Sigma_{2m}$. Let $s$ denote the number of indeterminates in this parametric set.

Our earlier result concerning polynomial ideals shows that there is a unique component $\mathcal{M}$ of the manifold of $\{F, F_1, \ldots, F_m\}_0$, regarded as an ideal of $\mathcal{F}[S, y_n^0, y_n^1, \ldots, y_n^{m+h}]$, which contains the zero $y_{ij} = \alpha_{ij}$ of this ideal. The dimension of $\mathcal{M}$ is $s$, for $s$ corresponds to $q$ of the earlier proof.

Now both $\Sigma_{1m}$ and $\Sigma_{2m}$ contain $\{F, F_1, \ldots, F_m\}_0$, while both have
the zero \( y_{ij} = \alpha_{ij} \). Hence their manifolds are in \( \mathcal{M} \). Since their manifolds are of dimension \( s \), however, they must coincide with \( \mathcal{M} \). Hence \( \Sigma_{1m} \) and \( \Sigma_{2m} \) are identical. But \( m \) was chosen so that \( \Sigma_{1m} \) contains a polynomial which is not in \( \Sigma_{2m} \), namely one of the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \). We have obtained a contradiction. This completes the proof of the theorem in the case that \( F \) is of equal order and effective order in \( y_n \).

If the order of \( F \) in \( y_n \) exceeds its effective order by \( d > 0 \), we replace each \( y_{nk} \) in \( F \) by \( z_{k-d} \), where \( z \) is a new indeterminate, and subscripts attached to \( z \) denote transforming. \( F \) goes into an irreducible polynomial \( \overline{F} \) which is of equal order and effective order in \( z \).

Evidently each component \( \mathcal{M} \) of the manifold of \( F \) corresponds to a unique component \( \mathcal{M} \) of the manifold of \( F \), and, conversely, each component of the manifold of \( F \) is obtained from a unique component of the manifold of \( \overline{F} \). The correspondence may be described as follows: each solution in \( \mathcal{M} \) is obtained from a solution in \( \mathcal{M} \) by leaving unchanged the elements assigned as values to \( y_1, \ldots, y_{n-1} \), and assigning to \( y_n \) an element whose \( d \)th transform is the element assigned as the value of \( z \) in \( \mathcal{M} \). This correspondence carries solutions common to two components of the manifold of \( F \) into solutions common to two components of the manifold of \( \overline{F} \). Solutions annulling the \( y_n \)-separant of \( F \) correspond to solutions annulling the \( z \)-separant of \( \overline{F} \).

The preceding proof shows that the theorem stated at the beginning of this note holds for \( \overline{F} \). The correspondence just described shows that its truth for \( \overline{F} \) implies its truth for \( F \). Hence it is true in general.

**References**


Rutgers University